CHBE320 LECTURE V

LAPLACE TRANSFORM AND TRANSFER FUNCTION

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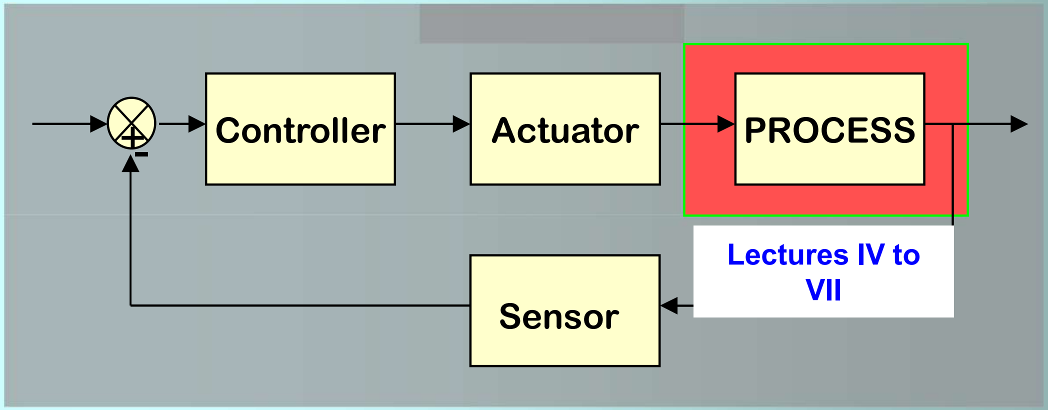
Road Map of the Lecture V

- **Laplace Transform and Transfer functions**
  - Definition of Laplace transform
  - Properties of Laplace transform
  - Inverse Laplace transform
  - Definition of transfer function
  - How to get the transfer functions
  - Properties of transfer function



SOLUTION OF LINEAR ODE

- **1st-order linear ODE**
  - **Integrating factor:** For $\frac{dx}{dt} + a(t)x = f(t)$, I.F. $= \exp(\int a(t)dt)$

$$[xe^{\int a(t)dt}]' = f(t)e^{\int a(t)dt} \longrightarrow x(t) = [\int f(t)e^{\int a(t)dt}\,dt + C]e^{-\int a(t)dt}$$

- **High-order linear ODE with constant coeffs.**
  - **Modes: roots of characteristic equation**

    For $a_2x'' + a_1x' + a_0x = f(t)$,

$$a_2p^2 + a_1p + a_0 = a_2(p - p_1)(p - p_2) = 0$$

  - **Depending on the roots, modes are**
    - **Distinct roots:** (e^{-p_1t}, e^{-p_2t})
    - **Double roots:** (e^{-p_1t}, te^{-p_1t})
    - **Imaginary roots:** $(e^{-\alpha t}\cos\beta\, t, e^{-\alpha}\ \sin\beta\, t)$

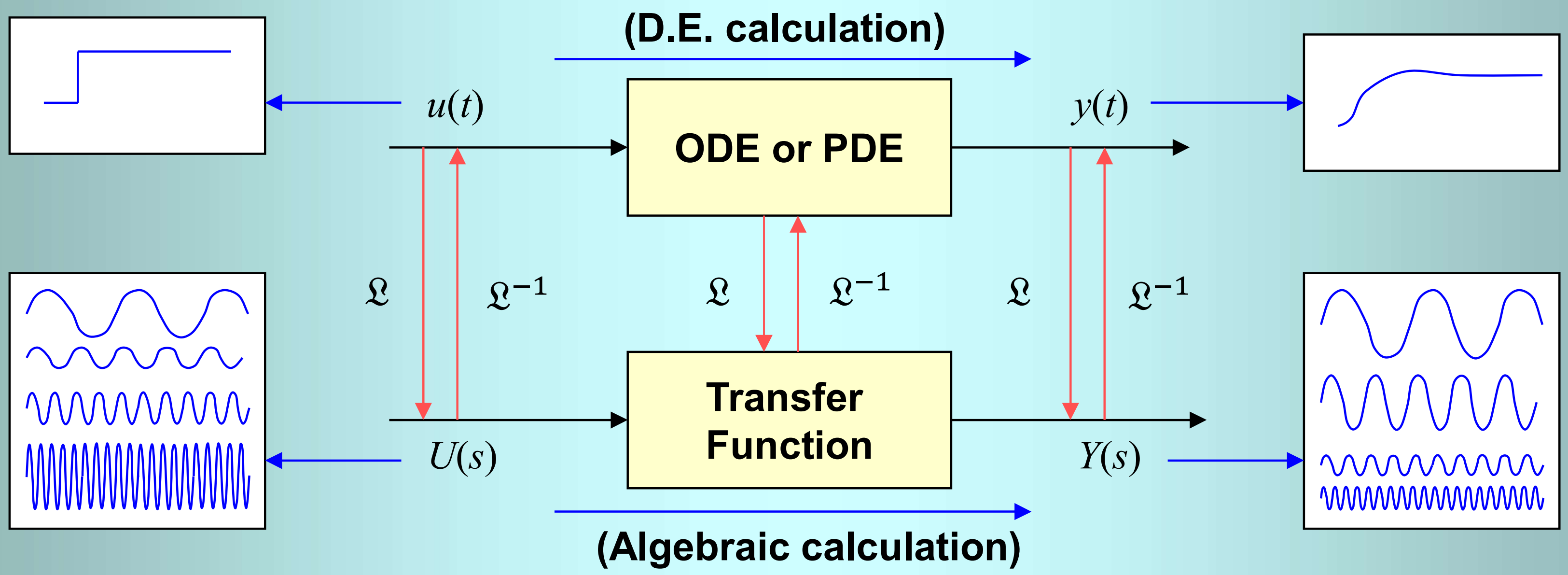
$\longrightarrow$

Solution is a linear combination of modes and the coefficients are decided by the initial conditions.

- **Many other techniques for different cases**

LAPLACE TRANSFORM FOR LINEAR ODE AND PDE

- **Laplace Transform**
  - **Not in time domain, rather in frequency domain**
  - **Derivatives and integral become some operators.**
  - **ODE is converted into algebraic equation**
  - **PDE is converted into ODE in spatial coordinate**
  - **Need inverse transform to recover time-domain solution**



DEFINITION OF LAPLACE TRANSFORM

- **Definition**

$$F(s) = \mathfrak{L}\{f(t)\} \triangleq \int_0^\infty f(t)e^{-st}dt$$

  - *F(s)* **is called** ***Laplace transform*** **of** *f(t)*.
  - *f(t)* **must be piecewise continuous.**
  - *F(s)* **contains no information on** *f(t)* **for** $t < 0$.
  - **The past information on** *f(t)* **(for** $t < 0$**) is irrelevant.**
  - **The** *s* **is a complex variable called "*Laplace transform variable*"**

- **Inverse Laplace transform**

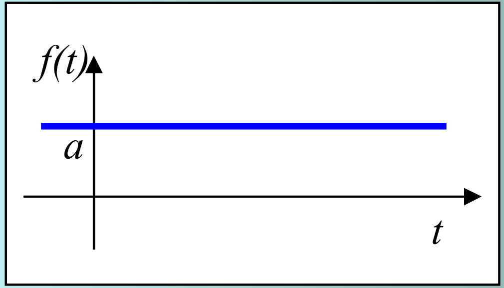
$$f(t) = \mathfrak{L}^{-1}\{F(s)\}$$

  - $\mathfrak{L}$ **and** $\mathfrak{L}^{-1}$ **are linear.** $\mathfrak{L}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

LAPLACE TRANSFORM OF FUNCTIONS

- **Constant function,** *a*

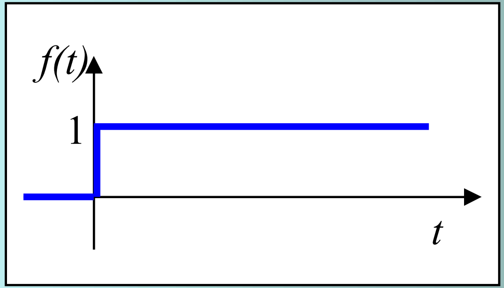
$$\mathfrak{L}\{a\} = \int_0^\infty a e^{-s}\,dt = -\frac{a}{s}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{a}{s}\right) = \frac{a}{s}$$



- **Step function,** S(*t*)

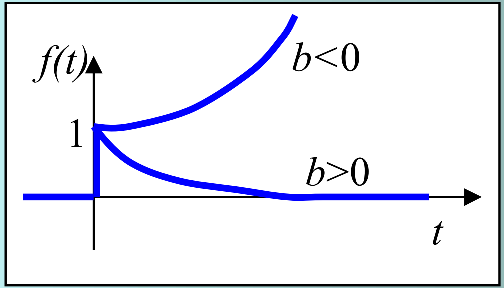
$$f(t) = \mathrm{S}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\mathfrak{L}\{\mathrm{S}(t)\} = \int_0^\infty e^{-s}\,dt = -\frac{1}{s}e^{-s}\Big|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

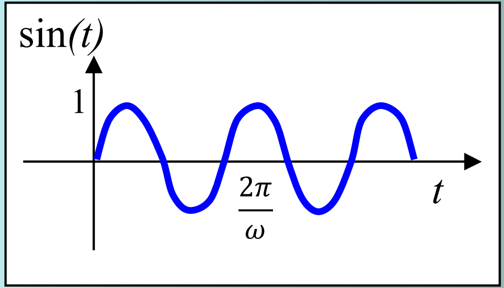


- **Exponential function,** e^{-bt}

$$\mathfrak{L}\{e^{-bt}\} = \int_0^\infty e^{-bt}e^{-s}\,dt = \frac{-1}{s+b}e^{-(b+s)t}\Big|_0^\infty = \frac{1}{s+b}$$



- **Trigonometric functions**
  - **Euler's Identity:** $e^{j\omega t} \triangleq \cos\omega t + j\sin\omega t$



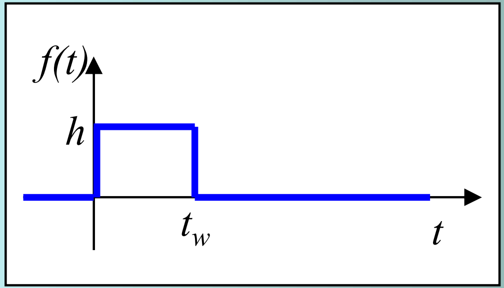
$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \qquad \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$\mathfrak{L}\{\sin\omega t\} = \mathfrak{L}\left\{\frac{1}{2j}e^{j\omega t}\right\} - \mathfrak{L}\left\{\frac{1}{2j}e^{-j\omega t}\right\} = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$\mathfrak{L}\{\cos\omega t\} = \mathfrak{L}\left\{\frac{1}{2}e^{j\omega t}\right\} + \mathfrak{L}\left\{\frac{1}{2}e^{-j\omega t}\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

- **Rectangular pulse, P(*t*)**

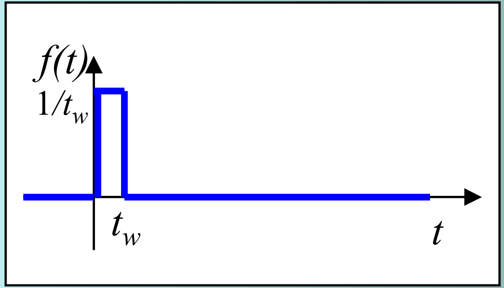
$$f(t) = \mathrm{P}(t) = \begin{cases} 0 & \text{for } t > t_w \\ h & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



$$\mathfrak{L}\{\mathrm{P}(t)\} = \int_0^{t_w} he^{-st}dt = -\frac{h}{s}e^{-st}\Big|_0^{t_w} = \frac{h}{s}\left(1 - e^{-t_w s}\right)$$

- **Impulse function,** $\delta(t)$

$$f(t)=\delta(t)=\lim_{t_w\to 0}\begin{cases}0 & \text{for } t>t_w\\ 1/t_w & \text{for } t_w\ge t\ge 0\\ 0 & \text{for } t<0\end{cases}$$



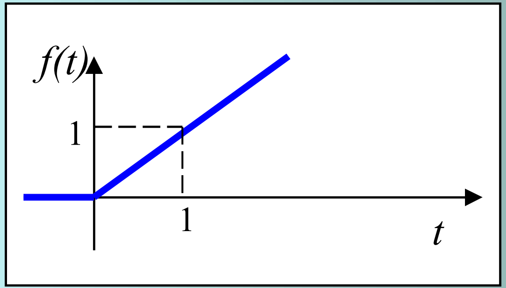
$$\mathfrak{L}\{\delta(t)\}=\lim_{t_w\to 0}\int_0^{t_w}\frac{1}{t_w}e^{-st}dt=\lim_{t_w\to 0}\frac{1}{t_w s}\left(1-e^{-t_w s}\right)=1$$

$$\left(\text{L}'\text{Hospital}'\text{s rule: }\lim_{t\to 0}\frac{f(t)}{g(t)}=\lim_{t\to 0}\frac{f'(t)}{g'(t)}\right)$$

- **Ramp function,** t

$$\mathfrak{L}\{t\}=\int_0^{\infty}te^{-st}dt$$

$$=\frac{t}{-s}e^{-s}\Big|_0^{\infty}-\int_0^{\infty}\frac{e^{-s}}{-s}dt=\frac{1}{s}\int_0^{\infty}e^{-s}\,dt=\frac{1}{s^2}$$



$$\left(\text{Integration by part: }\int_0^{\infty}f'\cdot g\,dt=f\cdot g\Big|_0^{\infty}-\int_0^{\infty}f\cdot g'dt\right)$$

- **Refer the Table 3.1 (Seborg et al.) for other functions**

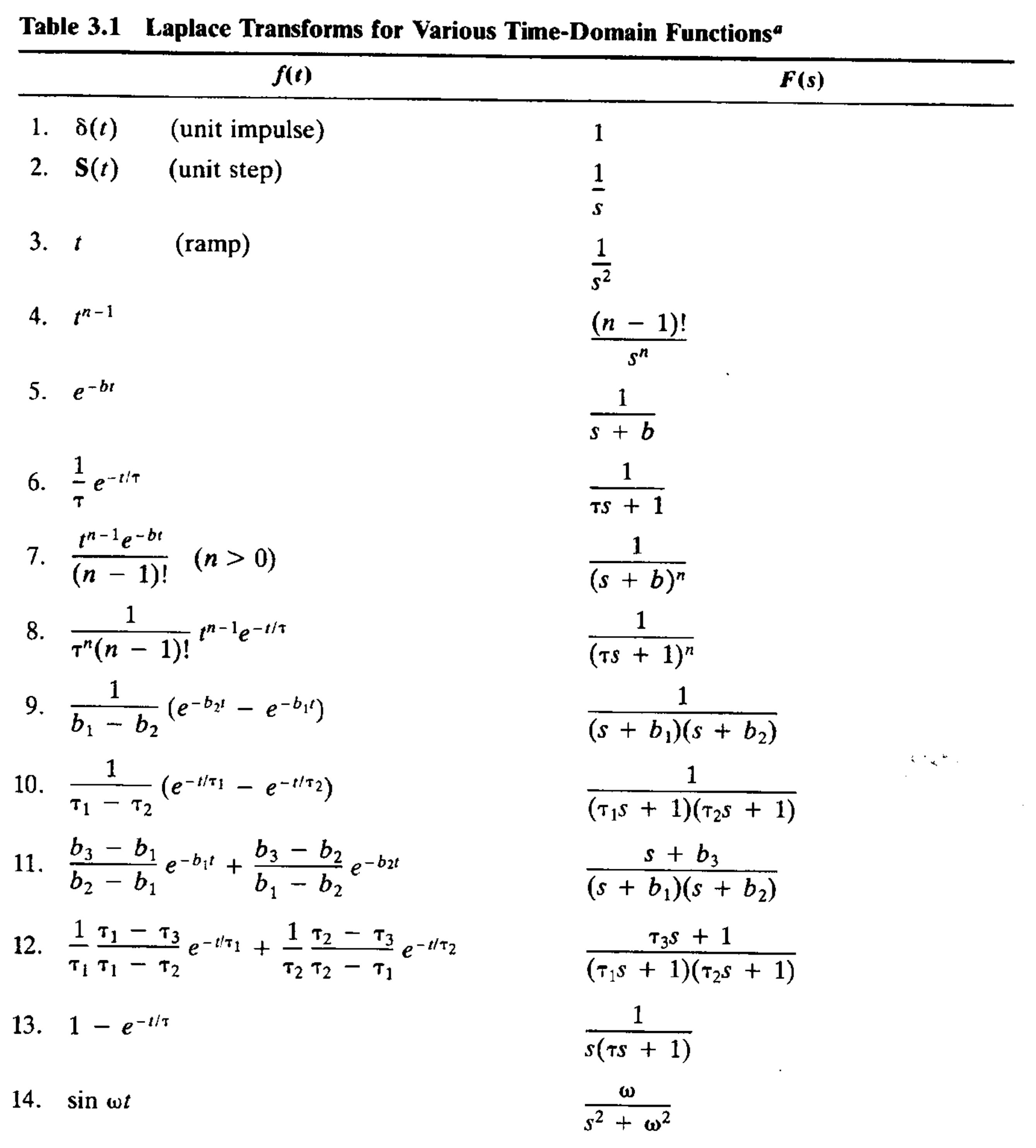


Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

	$f(t)$	$F(s)$
1.	$\delta(t)$ (unit impulse)	1
2.	$S(t)$ (unit step)	$\frac{1}{s}$
3.	t (ramp)	$\frac{1}{s^2}$
4.	t^{n-1}	$\frac{(n-1)!}{s^n}$
5.	e^{-bt}	$\frac{1}{s+b}$
6.	$\frac{1}{\tau} e^{-t/\tau}$	$\frac{1}{\tau s+1}$
7.	$\frac{t^{n-1}e^{-bt}}{(n-1)!} \quad (n > 0)$	$\frac{1}{(s+b)^n}$
8.	$\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$	$\frac{1}{(\tau s+1)^n}$
9.	$\frac{1}{b_1-b_2}(e^{-b_2 t} - e^{-b_1 t})$	$\frac{1}{(s+b_1)(s+b_2)}$
10.	$\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11.	$\frac{b_3-b_1}{b_2-b_1} e^{-b_1 t} + \frac{b_3-b_2}{b_1-b_2} e^{-b_2 t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12.	$\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1} e^{-t/\tau_2}$	$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13.	$1 - e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Table 3.1 (*Continued*)

	$f(t)$		$F(s)$
15.	$\cos \omega t$		$\dfrac{s}{s^2 + \omega^2}$
16.	$\sin (\omega t + \phi)$		$\dfrac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17.	$e^{-bt} \sin \omega t$	b, ω real	$\dfrac{\omega}{(s + b)^2 + \omega^2}$
18.	$e^{-bt} \cos \omega t$		$\dfrac{s + b}{(s + b)^2 + \omega^2}$
19.	$\dfrac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2}\, t/\tau)$ $(0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20.	$1 + \dfrac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$		$\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21.	$1 - \dfrac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1 - \zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1} \dfrac{\sqrt{1 - \zeta^2}}{\zeta}$ $(0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22.	$1 - e^{-\zeta t/\tau} [\cos(\sqrt{1 - \zeta^2}\, t/\tau)$ $+ \dfrac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\sqrt{1 - \zeta^2}\, t/\tau)]$ $(0 \le \lvert\zeta\rvert < 1)$		$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23.	$1 + \dfrac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \dfrac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$		$\dfrac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24.	$\dfrac{df}{dt}$		$sF(s) - f(0)$
25.	$\dfrac{d^n f}{dt^n}$		$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots$ $- s f^{(n-2)}(0) - f^{(n-1)}(0)$

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[a] Note that $f(t)$ and $F(s)$ are defined for $t \geqslant 0$ only.

PROPERTIES OF LAPLACE TRANSFORM

- **Differentiation**

$$\mathfrak{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty f' \cdot e^{-st} dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f \cdot (-s)e^{-st} dt \qquad (\text{by } i.b.p.)$$

$$= s\int_0^\infty f \cdot e^{-st} dt - f(0) = sF(s) - f(0)$$

$$\mathfrak{L}\left\{\frac{d^2f}{dt^2}\right\} = \int_0^\infty f'' \cdot e^{-s}\ dt = f(t)'e^{-s}\ \Big|_0^\infty - \int_0^\infty f' \cdot (-s)e^{-st} dt = s\int_0^\infty f' \cdot e^{-st} dt - f'(0)$$

$$= s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

$$\vdots$$

$$\mathfrak{L}\left\{\frac{d^nf}{dt^n}\right\} = \int_0^\infty f^{(n)} \cdot e^{-st} dt = f(t)^{(n-1)}e^{-st}\Big|_0^\infty - \int_0^\infty f^{(n-1)} \cdot (-s)e^{-st} dt$$

$$= s\int_0^\infty f^{(n-1)} \cdot e^{-st} dt - f^{(n-1)}(0) = s\left(\mathfrak{L}\left\{\frac{d^{n-1}f}{dt^{n-1}}\right\}\right) - f^{(n-1)}(0)$$

$$= s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- If $f(0) = f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0$,
  - Initial condition effects are vanished.
  - It is very convenient to use **deviation variables** so that all the effects of initial condition vanish.

$$\mathfrak{L}\left\{\frac{df}{dt}\right\} = sF(s)$$
$$\mathfrak{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s)$$
$$\vdots$$
$$\mathfrak{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s)$$

- Transforms of linear differential equations.

$$y(t)\xrightarrow{\mathfrak{L}}Y(s), \qquad u(t)\xrightarrow{\mathfrak{L}}U(s)$$
$$\frac{dy(t)}{dt}\xrightarrow{\mathfrak{L}}sY(s) \quad (\text{if } y(0)=0)$$

$$\tau\frac{dy(t)}{dt} = -y(t) + Ku(t) \;\; (y(0)=0)\xrightarrow{\mathfrak{L}}(\tau s+1)Y(s) = KU(s)$$

$$\frac{\partial T_L}{\partial t} = -v\frac{\partial T_L}{\partial z} + \frac{1}{\tau_{HL}}(T_w - T_L)\xrightarrow{\mathfrak{L}}\tau_{HL}v\frac{\partial \tilde{T}_L(s)}{\partial z} + (\tau_{HL}s+1)\tilde{T}_L(s) = \tilde{T}_w(s)$$

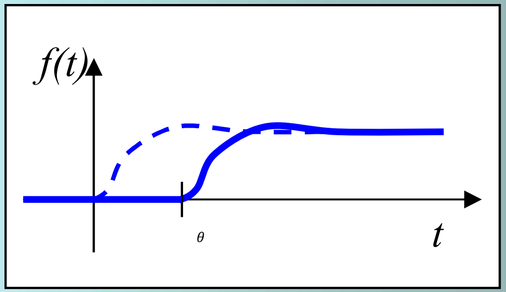
- **Integration**

$$\mathfrak{L}\left\{\int_0^t f(\xi)d\xi\right\} = \int_0^\infty \left(\int_0^t f(\xi)d\xi\right) e^{-st}dt$$

$$= \frac{e^{-st}}{-s}\int_0^t f(\xi)d\xi \Bigg|_0^{\infty}{}^{\mathbf{0}} + \frac{1}{s}\int_0^\infty f\cdot e^{-s}\,dt = \frac{F(s)}{s} \qquad (\text{by } i.b.p.)$$

$$\left(\text{Leibniz rule: } \frac{d}{dt}\int_{a(t)}^{b(t)} f(\tau)d\tau = f(b(t))\frac{db(t)}{dt} - f(a(t))\frac{da(t)}{dt}\right)$$

- **Time delay (Translation in time)**



$$f(t)\xrightarrow{+\theta\ \text{in}\,t} f(t-\theta)\,\mathrm{S}(t-\theta)$$

$$\mathfrak{L}\{f(t-\theta)\,\mathrm{S}(t-\theta)\} = \int_\theta^\infty f(t-\theta)e^{-st}dt = \int_0^\infty f(\tau)e^{-s(\tau+\theta)}d\tau \quad (\text{let } \tau = t-\theta)$$

$$= e^{-\theta s}\int_0^\infty f(\tau)e^{-\tau s}d\tau = e^{-\theta s}F(s)$$

- **Derivative of Laplace transform**

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f\cdot e^{-st}dt = \int_0^\infty f\cdot \frac{d}{ds}e^{-st}dt = \int_0^\infty (-t\cdot f)e^{-st}dt = \mathfrak{L}[-t\cdot f(t)]$$

- **Final value theorem**
  - **From the LT of differentiation, as** s **approaches to zero**

$$\lim_{s\to 0}\int_0^\infty \frac{df}{dt}\cdot e^{-s}\ dt = \int_0^\infty \frac{df}{dt}\cdot \lim_{s\to 0} e^{-s}\ \ dt = \lim_{s\to 0}[sF(s)-f(0)]$$

$$\int_0^\infty \frac{df}{dt}dt = f(\infty)-f(0) = \lim_{s\to 0} s\,F(s) - f(0) \Rightarrow \boxed{f(\infty) = \lim_{s\to 0} s\,F(s)}$$

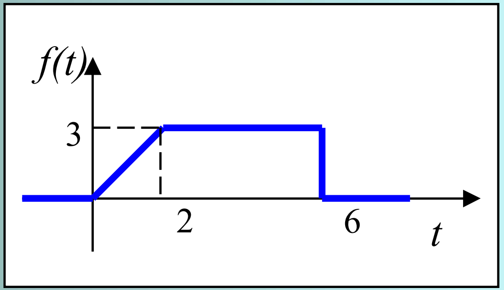
  - **Limitation:** $f(\infty)$ **has to exist. If it diverges or oscillates, this theorem is not valid.**

- **Initial value theorem**
  - **From the LT of differentiation, as** s **approaches to infinity**

$$\lim_{s\to \infty}\int_0^\infty \frac{df}{dt}\cdot e^{-s}\ dt = \lim_{s\to \infty}[sF(s)-f(0)]$$

$$\lim_{s\to \infty}\int_0^\infty \frac{df}{dt}e^{-s}\ dt = 0 = \lim_{s\to \infty} s\,F(s) - f(0) \Rightarrow \boxed{f(0) = \lim_{s\to \infty} s\,F(s)}$$

EXAMPLE ON LAPLACE TRANSFORM (1)

- 

$$f(t) = \begin{cases} 1.5t & \text{for } 0 \le t < 2 \\ 3 & \text{for } 2 \le t < 6 \\ 0 & \text{for } 6 \le t \\ 0 & \text{for } t < 0 \end{cases}$$

$$f(t) = 1.5t\,S(t) - 1.5(t-2)\,S(t-2) - 3\,S(t-6)$$

$$\therefore F(s) = \mathfrak{L}\{f(t)\} = \frac{1.5}{s^2}(1-e^{-2s}) - \frac{3}{s}e^{-6}$$

- For $F(s) = \dfrac{2}{s-5}$, find $f(0)$ and $f(\infty)$.
  - **Using the initial and final value theorems**

$$f(0) = \lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty}\frac{2s}{s-5} = 2$$

🚫

$$f(\infty) = \lim_{s\to 0} s\,F(s) = \lim_{s\to 0}\frac{2s}{s-5} = 0$$

  - **But the final value theorem is not valid because**

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 2\,e^{5t} = \infty$$

EXAMPLE ON LAPLACE TRANSFORM (2)

- **What is the final value of the following system?**

$$x'' + x' + x = \sin t\,;\ x(0) = x'(0) = 0$$

$$\Rightarrow\ s^2X(s) + sX(s) + X = \frac{1}{s^2+1}\ \Rightarrow\ \mathrm{x(s)} = \frac{1}{(s^2+1)(s^2+s+1)}$$

🚫

$$x(\infty) = \lim_{s\to 0}\frac{s}{(s^2+1)(s^2+s+1)} = 0$$

  - **Actually,** $x(\infty)$ **cannot be defined due to sin** ***t*** **term.**

- **Find the Laplace transform for** $(t\sin\omega\,t)$ **?**

$$\text{From }\frac{dF(s)}{ds} = \mathfrak{L}[-t\cdot f(t)]$$

$$\mathfrak{L}[t\cdot\sin\omega\,t] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

INVERSE LAPLACE TRANSFORM

- Used to recover the solution in time domain

$$\mathfrak{L}^{-1}\{F(s)\} = f(t)$$

  - From the table
  - By partial fraction expansion
  - By inversion using contour integral

$$f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\oint_C e^{st}F(s)ds$$

- Partial fraction expansion
  - After the partial fraction expansion, it requires to know some simple formula of inverse Laplace transform such as

$$\frac{1}{(\tau s+1)}, \frac{s}{(s+b)^2+\omega^2}, \frac{(n-1)!}{s^n}, \frac{e^{-\theta}}{\tau^2 s^2 + 2\zeta\tau s + 1}, \text{ etc.}$$

PARTIAL FRACTION EXPANSION

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)\cdots(s+p_n)} = \frac{\alpha_1}{(s+p_1)} + \cdots + \frac{\alpha_n}{(s+p_n)}$$

- **Case I: All p_i's are distinct and real**
  - **By a root-finding technique, find all roots (time-consuming)**
  - **Find the coefficients for each fraction**
    - **Comparison of the coefficients after multiplying the denominator**
    - **Replace some values for s and solve linear algebraic equation**
    - **Use of Heaviside expansion**
      - **Multiply both side by a factor, $(s+p_i)$, and replace s with $-p_i$.**

$$\alpha_i = (s+p_i)\frac{N(s)}{D(s)}\bigg|_{s=-p_i}$$

  - **Inverse LT:**

$$f(t) = \alpha_1 e^{-p_1 t} + \alpha_2 e^{-p_2 t} + \cdots + \alpha_n e^{-p_n t}$$

- **Case II: Some roots are repeated**

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p)^r}=\frac{b_{r-1}s^{r-1}+\cdots+b_0}{(s+p)^r}=\frac{\alpha_1}{(s+p)}+\cdots+\frac{\alpha_r}{(s+p)^r}$$

  - **Each repeated factors have to be separated first.**
  - **Same methods as Case I can be applied.**
  - **Heaviside expansion for repeated factors**

$$\alpha_{r-i}=\frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\Bigg|_{s=-p}\qquad (i=0,\cdots,r-1)$$

  - **Inverse LT**

$$f(t)=\alpha_1 e^{-pt}+\alpha_2 t e^{-pt}+\cdots+\frac{\alpha_r}{(r-1)!}t^{r-1}e^{-pt}$$

- **Case III: Some roots are complex**

$$F(s)=\frac{N(s)}{D(s)}=\frac{c_1 s+c_0}{s^2+d_1 s+d_0}=\frac{\alpha_1(s+b)+\beta_1\omega}{(s+b)^2+\omega^2}$$

  - **Each repeated factors have to be separated first.**
  - **Then,**

$$\frac{\alpha_1(s+b)+\beta_1\omega}{(s+b)^2+\omega^2}=\alpha_1\frac{(s+b)}{(s+b)^2+\omega^2}+\beta_1\frac{\omega}{(s+b)^2+\omega^2}$$

where $b=d_1/2,\qquad \omega=\sqrt{d_0-{d_1}^2/4}$

$$\alpha_1=c_1,\qquad \beta_1=(c_0-\alpha_1 b)/\omega$$

  - **Inverse LT**

$$f(t)=\alpha_1 e^{-b}\ \cos\omega t+\beta_1 e^{-bt}\sin\omega t$$

EXAMPLES ON INVERSE LAPLACE TRANSFORM

- $$F(s)=\frac{(s+5)}{s(s+1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s+3}\quad\text{(distinct)}$$

  - **Multiply each factor and insert the zero value**

$$\left.\frac{(s+5)}{(s+1)(s+2)(s+3)}\right|_{s=0}=\left.\left(A+s\frac{B}{s+1}+s\frac{C}{s+2}+s\frac{D}{s+3}\right)\right|_{s=0}\Rightarrow A=5/6$$

$$\left.\frac{(s+5)}{s(s+2)(s+3)}\right|_{s=-1}=\left.\left(\frac{A(s+1)}{s}+B+\frac{C(s+1)}{s+2}+\frac{D(s+1)}{s+3}\right)\right|_{s=-1}\Rightarrow B=-2$$

$$\left.\frac{(s+5)}{s(s+1)(s+3)}\right|_{s=-2}=\left.\left(\frac{A(s+2)}{s}+\frac{B(s+2)}{s+1}+C+\frac{D(s+2)}{s+3}\right)\right|_{s=-2}\Rightarrow C=3/2$$

$$\left.\frac{(s+5)}{s(s+1)(s+2)}\right|_{s=-3}=\left.\left(\frac{A(s+3)}{s}+\frac{B(s+3)}{s+1}+\frac{C(s+3)}{s+2}+D\right)\right|_{s=-3}\Rightarrow D=-1/3$$

$$\therefore f(t)=\mathfrak{L}^{-1}\{F(s)\}=\frac{5}{6}-2e^{-t}+\frac{3}{2}e^{-2t}-\frac{1}{3}e^{-3t}$$

- $$F(s) = \frac{1}{(s+1)^3(s+2)} = \frac{As^2+Bs+C}{(s+1)^3} + \frac{D}{(s+2)} \quad \text{(repeated)}$$

$$\begin{aligned} 1 &= (As^2+Bs+C)(s+2) + D(s+1)^3 \\ &= (A+D)s^3 + (2A+B+3D)s^2 + (2B+C+3D)s + (2C+D) \end{aligned}$$

$$\therefore A = -D, \qquad 2A+B+3D=0, \qquad 2B+C+3D=0, \qquad 2C+D=1$$

$$\Rightarrow A=1, \qquad B=1, \qquad C=1, \qquad D=-1$$

  - **Use of Heaviside expansion** $\alpha_{r-i} = \frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\Bigg|_{s=-p} \quad (i = 0, \cdots, r-1)$

$$\frac{s^2+s+1}{(s+1)^3} = \frac{\alpha_1}{(s+1)} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{(s+1)^3}$$

$$(i=0):\ \alpha_3 = (s^2+s+1)\Big|_{s=-1} = 1$$

$$(i=1):\ \alpha_2 = \frac{1}{1!}\frac{d}{ds}(s^2+s+1)\Bigg|_{s=-1} = -1$$

$$(i=2):\ \alpha_1 = \frac{1}{2!}\frac{d^2}{ds^2}(s^2+s+1)\Bigg|_{s=-1} = 1$$

$$\therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} = e^{-t} - te^{-t} + \frac{1}{2}t^2e^{-t} - e^{-2}$$

- $$F(s) = \frac{(s+1)}{s^2(s^2+4s+5)} = \frac{A(s+2)+B}{(s+2)^2+1} + \frac{Cs+D}{s^2} \quad \text{(complex)}$$

$$\begin{aligned} s+1 &= A(s+2)s^2 + Bs^2 + (Cs+D)(s^2+4s+5) \\ &= (A+C)s^3 + (2A+B+4C+D)s^2 + (5C+4D)s + 5D \end{aligned}$$

$$\therefore A = -C, \qquad 2A+B+4C+D = 0, \qquad 5C+4D = 1, \qquad 5D = 1$$

$$\Rightarrow A = -1/25, \qquad B = -7/25, \qquad C = 1/25, \qquad D = 1/5$$

$$\frac{A(s+2)+B}{(s+2)^2+1} = -\frac{1}{25}\frac{(s+2)}{(s+2)^2+1} - \frac{7}{25}\frac{B}{(s+2)^2+1}$$

$$\frac{Cs+D}{s^2} = \frac{1}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2}$$

$$\therefore f(t) = \mathfrak{L}^{-1}\{F(s)\} = -\frac{1}{25}e^{-2t}\cos t - \frac{7}{25}e^{-2}\ \sin t + \frac{1}{25} + \frac{1}{5}t$$

- $$F(s)=\frac{1+e^{-2s}}{(4s+1)(3s+1)}=\left(\frac{A}{4s+1}+\frac{B}{3s+1}\right)(1+e^{-2}\ )\quad \text{(Time delay)}$$

$$A=1/(3s+1)\Big|_{s=-1/4}=4,\qquad B=1/(4s+1)\Big|_{s=-1/3}=-3$$

$$\therefore f(t)=\mathfrak{L}^{-1}\{F(s)\}=\mathfrak{L}^{-1}\left\{\frac{4}{4s+1}-\frac{3}{3s+1}\right\}+\mathfrak{L}^{-1}\left\{\frac{4e^{-2s}}{4s+1}-\frac{3e^{-2}}{3s+1}\right\}$$
$$=e^{-t/4}-e^{-t/3}+\left(e^{-(t-2)/4}-e^{-(t-2)/3}\right)S(t-2)$$

SOLVING ODE BY LAPLACE TRANSFORM

- **Procedure**
  1. **Given linear ODE with initial condition,**
  2. **Take Laplace transform and solve for output**
  3. **Inverse Laplace transform**

- **Example:**

Solve for $5\dfrac{dy}{dt} + 4y = 2;\ y(0) = 1$

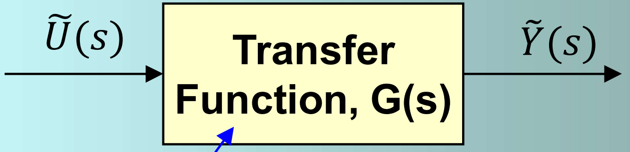
$$\mathfrak{L}\left\{5\frac{dy}{dt}\right\} + \mathfrak{L}\{4y\} = \mathfrak{L}\{2\} \;\Rightarrow\; 5(sY(s) - y(0)) + 4Y(s) = \frac{2}{s}$$

$$(5s+4)Y(s) = \frac{2}{s} + 5 \;\Rightarrow\; Y(s) = \frac{5s+2}{s(5s+4)}$$

$$\therefore y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \mathfrak{L}^{-1}\left\{\frac{0.5}{s} + \frac{2.5}{5s+4}\right\} = 0.5 + 0.5e^{-0.8t}$$

TRANSFER FUNCTION (1)

- **Definition**
  - An algebraic expression for the dynamic relation **between the input and output** of the process model



$$5\frac{dy}{dt} + 4y = u;\ y(0) = 1$$

Let $\tilde{y} = y - 1$ and $\tilde{u} = u - 4$

$$(5s+4)\tilde{Y}(s) = \tilde{U}(s) \quad \Rightarrow \quad \frac{\tilde{Y}(s)}{\tilde{U}(s)} = \frac{1}{5s+4} = \frac{0.25}{1.25s+1} = G(s)$$

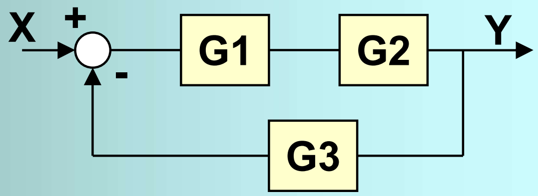
- **How to find transfer function**
  1. Find the **equilibrium point**
  2. If the system is nonlinear, then **linearize** around equil. point
  3. Introduce **deviation variables**
  4. Take **Laplace transform** and solve for output
  5. Do the **Inverse Laplace transform** and recover the original variables from deviation variables

TRANSFER FUNCTION (2)

- **Benefits**
  - Once TF is known, the output response to various given inputs can be obtained easily.

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \mathfrak{L}^{-1}\{G(s)U(s)\} \neq \mathfrak{L}^{-1}\{G(s)\}\mathfrak{L}^{-1}\{U(s)\}$$

  - Interconnected system can be analyzed easily.
    - By block diagram algebra



$$\frac{Y(s)}{X(s)} = \frac{G1(s)G2(s)}{1 + G1(s)G2(s)G3(s)}$$

  - Easy to analyze the qualitative behavior of a process, such as stability, speed of response, oscillation, etc.
    - By inspecting “Poles” and “Zeros”
    - Poles: all *s*’s satisfying $D(s)=0$
    - Zeros: all *s*’s satisfying $N(s)=0$

TRANSFER FUNCTION (3)

- **Steady-state Gain**: **The ratio between ultimate changes in input and output**

$$\text{Gain}=K=\frac{\Delta\text{ouput}}{\Delta\text{input}}=\frac{(y(\infty)-y(0))}{(u(\infty)-u(0))}$$

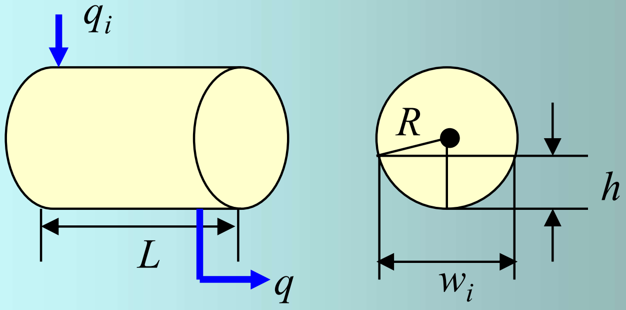
  - **For a unit step change in input, the gain is the change in output**
  - **Gain may not be definable: for example, integrating processes and processes with sustaining oscillation in output**
  - **From the final value theorem, unit step change in input with zero initial condition gives**

$$K=\frac{y(\infty)}{1}=\lim_{s\to 0} s\,Y(s)=\lim_{s\to 0} s\,G(s)\frac{1}{s}=\lim_{s\to 0} G\,(s)$$

  - **The transfer function itself is an impulse response of the system** $Y(s)=G(s)U(s)=G(s)\mathfrak{L}\{\delta(\text{t})\}=G(s)$

EXAMPLE

- **Horizontal cylindrical storage tank (Ex4.7)**



$$\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho q_i - \rho q$$

$$V(h) = \int_0^h L w_i(\tilde{h}) d\tilde{h} \Rightarrow \frac{dV}{dt} = L w_i(h) \frac{dh}{dt}$$

$$w_i(h)/2 = \sqrt{R^2 - (R-h)^2} = \sqrt{(2R-h)h}$$

$$w_i L \frac{dh}{dt} = q_i - q \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{2L\sqrt{(D-h)h}}(q_i - q) \quad \text{(Nonlinear ODE)}$$

  - **Equilibrium point:** $(\bar{q}_i, \bar{q}, \bar{h})$ $\qquad 0 = (\bar{q}_i - \bar{q})\,/(2L\sqrt{(D-\bar{h})\bar{h}})$

    (if $\bar{q}_i = \bar{q}$, $\bar{h}$ can be any value in $0 \le \bar{h} \le D$.)

  - **Linearization:**

$$\frac{dh}{dt} = f(h, q_i, q) = \left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})} (h - \bar{h}) + \left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})} (q_i - \bar{q}_i) + \left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})} (q - \bar{q})$$

$$\left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = (\bar{q}_i - \bar{q})\frac{\partial}{\partial h}\frac{-1}{2L\sqrt{(D-h)h}} = 0 \quad (\because \bar{q}_i = \bar{q})$$

Let this term be k

$$\left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = \frac{-1}{2L\sqrt{(D-\bar{h})\bar{h}}}, \qquad \left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = \frac{1}{2L\sqrt{(D-\bar{h})\bar{h}}}$$

$$s\tilde{H}(s) = k\tilde{Q}_i(s) - k\tilde{Q}(s)$$

- **Transfer function between** $\tilde{H}(s)$ and $\tilde{Q}(s)$: $-\dfrac{k}{s}$ **(integrating)**
- **Transfer function between** $\tilde{H}(s)$ and $\tilde{Q}_i(s)$: $\dfrac{k}{s}$ **(integrating)**

- **If $\bar{h}$ is near 0 or D, k becomes very large and $\bar{h}$ is around *D*/2, k becomes minimum.**

$\Rightarrow$ **The model could be quite different depending on the operating condition used for the linearization.**

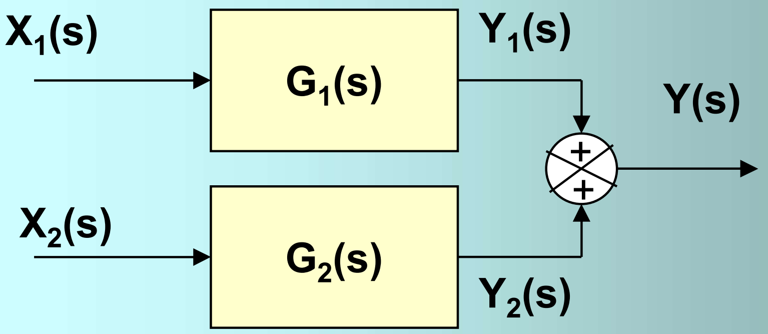
$\Rightarrow$ **The best suitable range for the linearization in this case is around *D*/2. (less change in gain)**

$\Rightarrow$ **Linearized model would be valid in very narrow range near 0.**

PROPERTIES OF TRANSFER FUNCTION

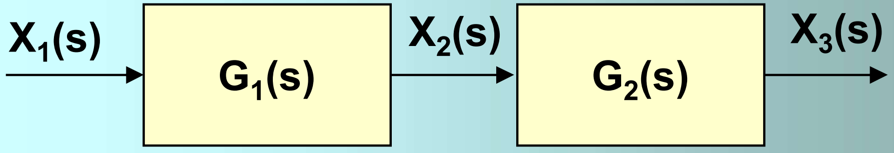
- **Additive property**

$$\begin{aligned} Y(s) &= Y_1(s) + Y_2(s) \\ &= G_1(s)X_1(s) + G_2(s)X_2(s) \end{aligned}$$



- **Multiplicative property**

$$\begin{aligned} X_3(s) &= G_2(s)X_2(s) \\ &= G_2(s)[G_1(s)X_1(s)] \end{aligned}$$



- **Physical realizability**
  - In a transfer function, the order of numerator(m) is greater than that of denominator(n): called "physically unrealizable"
  - The order of derivative for the input is higher than that of output. (requires future input values for current output)

EXAMPLES ON TWO TANK SYSTEM

- **Two tanks in series (Ex3.7)**
  - **No reaction**

$$V_1\frac{dc_1}{dt}+qc_1=qc_i$$

$$V_2\frac{dc_2}{dt}+qc_2=qc_1$$

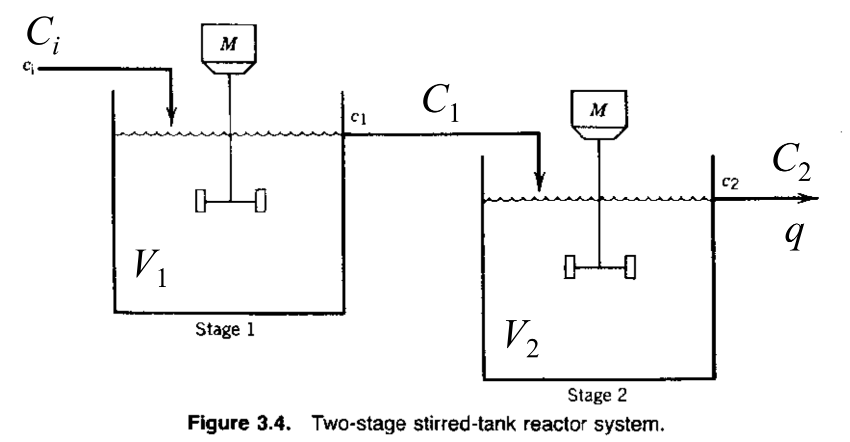


Figure 3.4. Two-stage stirred-tank reactor system.

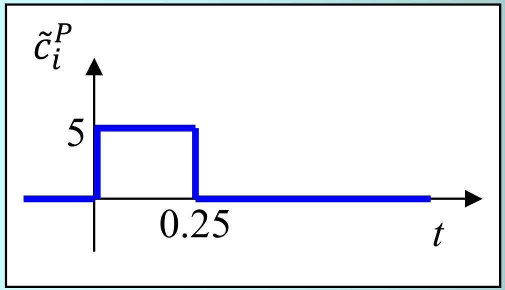
  - **Initial condition:** $c_1(0)=c_2(0)=1$ kg mol/m^3 **(Use deviation var.)**
  - **Parameters:** V_1/q**=2 min.,** V_2/q**=1.5 min.**
  - **Transfer functions**

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)}=\frac{1}{(V_1/q)s+1} \qquad \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}=\frac{1}{(V_2/q)s+1}$$

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)}=\frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)}=\frac{1}{((V_2/q)s+1)((V_1/q)s+1)}$$

- **Pulse input**

$$\tilde{C}_i^P(s) = \frac{5}{s}\left(1 - e^{-0.25s}\right)$$



- **Equivalent impulse input**

$$\tilde{C}_i^\delta(s) = \mathfrak{L}\{(5 \times 0.25)\delta(t)\} = 1.25$$

- **Pulse response vs. Impulse response**

$$\tilde{C}_1^P(s) = \frac{1}{2s+1}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)}\left(1 - e^{-0.25s}\right)$$

$$= \left(\frac{5}{s} - \frac{10}{2s+1}\right)\left(1 - e^{-0.25s}\right)$$

$$\Rightarrow \tilde{c}_1^P(t) = 5\left(1 - e^{-t/2}\right) - 5\left(1 - e^{-(t-0.25)/2}\right)S(t - 0.25)$$

$$\tilde{C}_1^\delta(s) = \frac{1}{2s+1}\tilde{C}_i^\delta(s) = \frac{1.25}{(2s+1)}$$

$$\Rightarrow \tilde{c}_1^\delta = 0.625e^{-t/2}$$

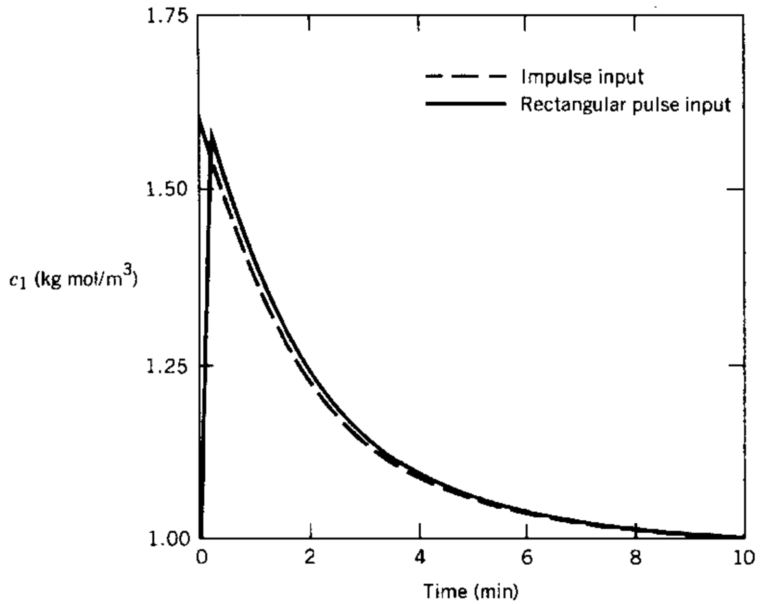


Figure 3.6. Reactor Stage 1 response.

$$\tilde{C}_2^P(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)(1.5s+1)}\left(1-e^{-0.25s}\right)$$

$$= \left(\frac{5}{s} - \frac{40}{2s+1} + \frac{22.5}{1.5s+1}\right)\left(1-e^{-0.25s}\right)$$

$$\Rightarrow \tilde{c}_2^P(t) = \left(5 - 20e^{-t/2} + 15e^{-t/1.5}\right) - \left(5 - 20e^{-(t-0.25)/2} + 15e^{-(t-0.25)/1.5}\right) S(t-0.25)$$

$$\tilde{C}_2^\delta(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^\delta(s)$$

$$= \frac{1.25}{(2s+1)(1.5s+1)}$$

$$= \frac{5}{2s+1} - \frac{3.75}{1.5s+1}$$

$$\Rightarrow \tilde{c}_2^\delta = 2.5e^{-t/2} - 2.5e^{-t/1.5}$$

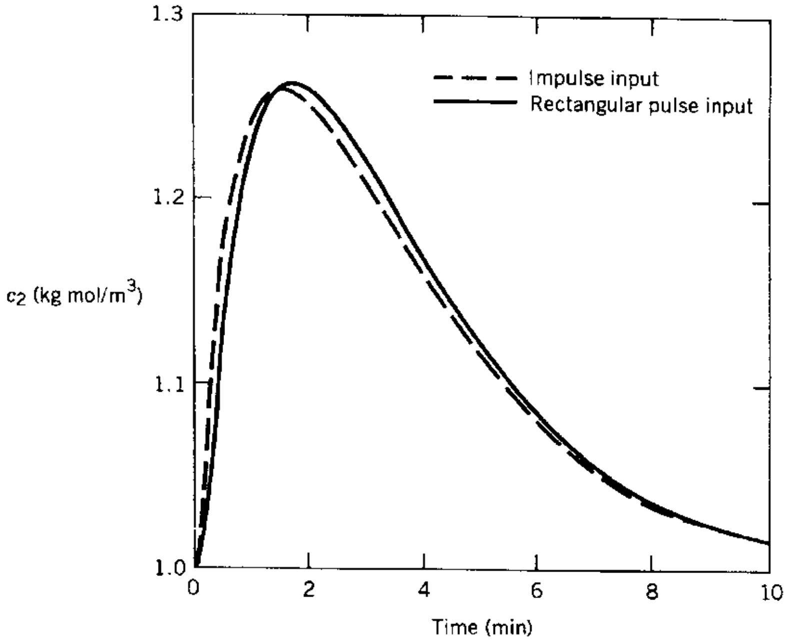


Figure 3.7. Reactor Stage 2 response.