CHBE320 LECTURE VI
DYNAMIC BEHAVIORS OF REPRESENTATIVE PROCESSES

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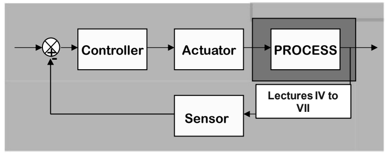
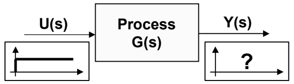
Road Map of the Lecture VI

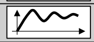
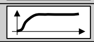
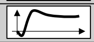
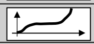
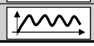
- **Dynamic Behavior of Representative Processes**
  - Open-loop responses
    - Step input
    - Impulse input
    - Sinusoidal input
    - Ramp input
  - Bode diagram analysis
  - Effect of pole/zero location



REPRESENTATIVE TYPES OF RESPONSE

- **For step inputs**



Y(t)	Type of Model, G(s)
	Nonzero initial slope, no overshoot or nor oscillation, 1st order model
	1st order+Time delay
	Underdamped oscillation, 2nd or higher order
	Overdamped oscillation, 2nd or higher order
	Inverse response, negative (RHP) zeros
	Unstable, no oscillation, real RHP poles
	Unstable, oscillation, complex RHP poles
	Sustained oscillation, pure imaginary poles

1ST ORDER SYSTEM

- **First-order linear ODE** (assume all deviation variables)

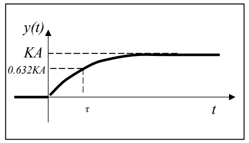
$$\tau\frac{dy(t)}{dt} = -y(t) + Ku(t) \xrightarrow{\mathfrak{L}} (\tau s + 1)Y(s) = KU(s)$$

- **Transfer function:** $\dfrac{Y(s)}{U(s)} = \dfrac{K}{(\tau s + 1)}$ → Gain, Time constant

- **Step response:**

With $U(s) = A/s$,

$$Y(s) = \frac{KA}{s(\tau s + 1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) = KA(1 - e^{-t/\tau})$$



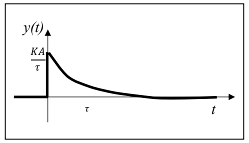
– $y(\tau) = KA(1 - e^{-\tau/\tau}) \approx 0.632KA$

– $KA(1 - e^{-t/\tau}) \geq 0.99KA \Rightarrow t \approx 4.6\tau$ (Settling time=4τ~5τ)

– $y'(0) = KAe^{-t/\tau}/\tau\big|_{t=0} = KA/\tau \neq 0$ (Nonzero initial slope)

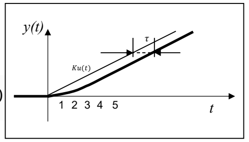
- **Impulse response**

With $U(s) = A$,
$$Y(s) = \frac{KA}{(\tau s + 1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) = \frac{KA}{\tau} e^{-t/\tau}$$



- **Ramp response**

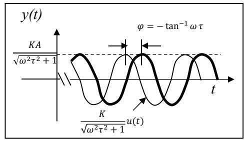
With $U(s) = a/s^2$,
$$Y(s) = \frac{Ka}{s^2(\tau s + 1)} \xrightarrow{\mathfrak{L}^{-1}} y(t) = Ka\tau e^{-t/\tau} + Ka(t - \tau)$$



- **Sinusoidal response**

With $U(s) = \mathfrak{L}[A \sin \omega t] = A\omega/(s^2 + \omega^2)$,
$$Y(s) = \frac{KA\omega}{(\tau s + 1)(s^2 + \omega^2)} \xrightarrow{\mathfrak{L}^{-1}}$$

$$y(t) = \frac{KA}{\omega^2 \tau^2 + 1}(\omega\tau e^{-t/\tau} - \omega\tau \cos \omega t + \sin \omega t)$$



- **Ultimate sinusoidal response** $(t \to \infty)$

$$y_\infty(t) = \lim_{t\to\infty} \frac{KA}{\omega^2\tau^2 + 1}(\cancelto{0}{\omega\tau e^{-t/\tau}} - \omega\tau \cos \omega t + \sin \omega t)$$
$$= \frac{KA}{\omega^2\tau^2 + 1}(-\omega\tau \cos \omega t + \sin \omega t)$$
$$= \underbrace{\frac{KA}{\sqrt{\omega^2\tau^2 + 1}}}_{\text{Amplitude}} \sin(\omega t + \underbrace{\varphi}_{\text{Phase angle}}) \qquad (\varphi = -\tan^{-1} \omega \tau)$$

– **The output has the same period of oscillation as the input.**

– **But the amplitude is attenuated and the phase is shifted.**

Normalized Amplitude Ratio (AR_N) $= \dfrac{1}{\sqrt{\omega^2\tau^2 + 1}} < 1$ **Phase angle** $= -\tan^{-1} \omega \tau$

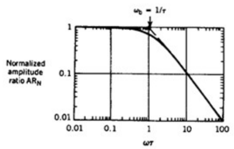
– **High frequency input will be attenuated more and phase is shifted more.**

BODE PLOT FOR 1ST ORDER SYSTEM

- **AR plot asymptote**

$$AR_N(\omega \to 0) = \lim_{\omega\to 0} \frac{1}{\sqrt{\omega^2\tau^2 + 1}} = 1$$

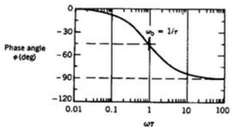
$$AR_N(\omega \to \infty) = \lim_{\omega\to \infty} \frac{1}{\sqrt{\omega^2\tau^2 + 1}} = \frac{1}{\omega\tau}$$



- **Phase plot asymptote**

$$\varphi(\omega \to 0) = -\lim_{\omega\to 0} \tan^{-1} \omega \tau = 0^\circ$$

$$\varphi(\omega \to \infty) = -\lim_{\omega\to \infty} \tan^{-1} \omega \tau = -90^\circ$$



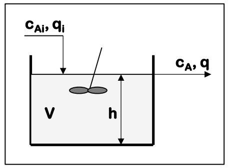
- **It is also called "low-pass filter"**

1ST ORDER PROCESSES

- **Continuous Stirred Tank**

$$V \frac{dc_A}{dt} = qc_{Ai} - qc_A$$

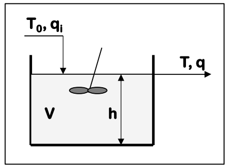
$$\frac{C_A(s)}{C_{Ai}(s)} = \frac{q}{Vs + q} = \frac{1}{(V/q)s + 1}$$



– **With constant heat capacity and density**

$$\rho V C_p \frac{d(T - T_{ref})}{dt} = \rho q C_p (T_0 - T_{ref}) - \rho q C_p (T - T_{ref})$$

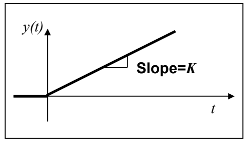
$$\frac{T(s)}{T_0(s)} = \frac{q}{Vs + q} = \frac{1}{(V/q)s + 1}$$



INTEGRATING SYSTEM

- $\frac{dy(t)}{dt} = Ku(t) \xrightarrow{\mathfrak{L}} sY(s) = KU(s)$
- **Transfer Function:** $\frac{Y(s)}{U(s)} = \frac{K}{s}$
- **Step Response**

With $U(s) = 1/s$,
$Y(s) = \frac{K}{s^2} \xrightarrow{\mathfrak{L}^{-1}} y(t) = Kt$



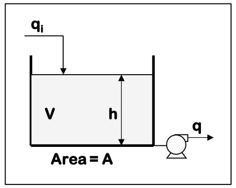
  - **The output is an integration of input.**
  - **Impulse response is a step function.**
  - **Non self-regulating system**

INTEGRATING PROCESSES

- **Storage tank with constant outlet flow**
  - **Outlet flow is pumped out by a constant-speed, constant-volume pump**
  - **Outlet flow is not a function of head.**

$$A\frac{dh}{dt} = q_i - q$$

$$\frac{H(s)}{Q_i(s)} = \frac{1}{As} \qquad \frac{H(s)}{Q(s)} = -\frac{1}{As}$$



2ND ORDER SYSTEM

- **2nd order linear ODE**

$$\tau^2\frac{d^2y(t)}{dt^2} + 2\zeta\tau\frac{dy(t)}{dt} + y(t) = Ku(t) \xrightarrow{\mathfrak{L}} (\tau^2s^2 + 2\zeta\tau s + 1)Y(s) = KU(s)$$

- **Transfer Function:**

$$\frac{Y(s)}{U(s)} = \frac{K}{(\tau^2s^2 + 2\zeta\tau s + 1)}$$

Gain
Time constant
Damping Coefficient

- **Step response**
  - **Varies with the type of roots of denominator of the TF.**
    - **Real part of roots should be negative for stability:** $\zeta \geq 0$
    - **Two distinct real roots (** $\zeta > 1$ **): overdamped (no oscillation)**
    - **Double root (** $\zeta = 1$**): critically damped (no oscillation)**
    - **Complex roots (** $0 \leq \zeta < 1$ **): underdamped (oscillation)**

- **Case I** $(\zeta > 1)$ **with** $U(s)=1/s$

$$Y(s) = \frac{K}{s(\tau^2s^2 + 2\zeta\tau s + 1)} = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)} \xrightarrow{\mathfrak{L}^{-1}} \boxed{y(t) = K\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{(\tau_1 - \tau_2)}\right)}$$

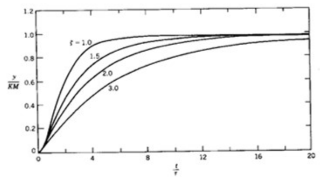
- **Case II** $(\zeta = 1)$

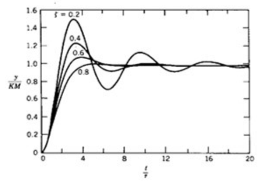
$$Y(s) = \frac{K}{s(\tau^2s^2 + 2\tau s + 1)} = \frac{K}{s(\tau s + 1)^2} \xrightarrow{\mathfrak{L}^{-1}} \boxed{y(t) = K\left[1 - (1 + t/\tau)e^{-t/\tau}\right]}$$

- **Case III** $(0 \leq \zeta < 1)$

Natural frequency

$$Y(s) = \frac{K}{s(\tau^2s^2 + 2\zeta\tau s + 1)} \xrightarrow{\mathfrak{L}^{-1}} \boxed{y(t) = K\left[1 - e^{-\zeta t/\tau}\left\{\cos\alpha t + \frac{\zeta}{\alpha\tau}\sin\alpha t\right\}\right] \quad \left(\alpha = \frac{\sqrt{1-\zeta^2}}{\tau}\right)}$$





- **Ultimate sinusoidal response**

With $U(s) = \mathfrak{L}[A \sin \omega t]$,

$$Y(s) = \frac{KA\omega}{(\tau^2 s^2 + 2\zeta\tau s + 1)(s^2 + \omega^2)} \xrightarrow{\mathfrak{L}^{-1}}$$

$$y(t) = \frac{KA}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}} \sin(\omega t + \varphi) \qquad (\varphi = -\tan^{-1}\frac{2\zeta\omega\tau}{1-\omega^2\tau^2})$$

– **Other method to find ultimate sinusoidal response**

For $(s + \alpha + j\omega)$, $y(t)$ has $e^{-(\alpha + j\omega)t}$ and it becomes $e^{-j\omega t}$ as $t \to \infty$ $(\alpha > 0)$.

$$G(s) = \frac{K}{(\tau^2 s^2 + 2\zeta\tau s + 1)} \xrightarrow{s \to j\omega} G(j\omega) = \frac{K}{(1-\tau^2\omega^2) + 2j\zeta\tau\omega}$$

$$AR = |G(j\omega)| = \left|\frac{K}{(1-\tau^2\omega^2) + j\tau\omega}\right| = \frac{K}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\varphi = \measuredangle G(j\omega) = \tan^{-1}\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))} = -\tan^{-1}\frac{2\zeta\omega\tau}{1-\omega^2\tau^2}$$

BODE PLOT FOR 2ND ORDER SYSTEM

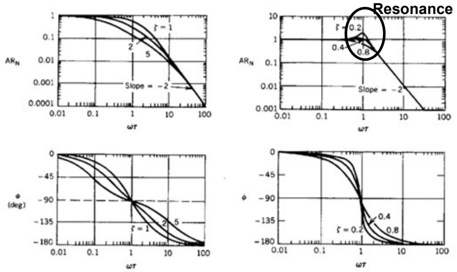
- **AR plot** $AR_N(\omega \to \infty) = \lim_{\omega\to\infty} \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}} = \frac{1}{(\omega\tau)^2}$
- **Phase plot** $\varphi(\omega \to \infty) = -\lim_{\omega\to\infty} \tan^{-1}\frac{2\zeta\omega\tau}{1-\omega^2\tau^2} = \lim_{\omega\to\infty}\tan^{-1}\frac{-2\zeta}{-\omega\tau} = -180^{\circ}$
- **Resonance**

$d(AR_N)/d\omega = 0$

$$\omega_{max} = \frac{\sqrt{1-2\zeta^2}}{\tau}$$

for $0 < \zeta < 0.707$

The amplitude of output oscillation is bigger than that of input when the resonance occurs .



1ST ORDER VS. 2ND ORDER (OVERDAMPED)

- **Initial slope of step response**

1st order: $y'(0) = \lim_{s\to\infty}\{s^2 Y(s)\} = \lim_{s\to\infty}\frac{KAs}{\tau s + 1} = \frac{KA}{\tau} \neq 0$

2nd order: $y'(0) = \lim_{s\to\infty}\{s^2 Y(s)\} = \lim_{s\to\infty}\frac{KAs}{\tau^2 s^2 + 2\zeta\tau s + 1} = 0$

- **Shape of the curve (Convexity)**

1st order: $y''(t) = -(KA/\tau^2)e^{-t/\tau} < 0$ (For $K > 0$) $\Rightarrow$ No inflection

2nd order: $y''(t) = -\frac{KA}{\tau_1 - \tau_2}(\frac{e^{-t/\tau_1}}{\tau_1} - \frac{e^{-t/\tau_2}}{\tau_2})$

$(+ \to -$ as $t \uparrow)$ $\Rightarrow$ Inflection

CHARACTERIZATION OF SECOND ORDER SYSTEM

- **2nd order Underdamped response**
  – **Rise time (t_r)**

  $t_r = \tau(n\pi - \cos^{-1}\zeta)/\sqrt{1-\zeta^2}$ $(n = 1)$

  – **Time to 1st peak (t_p)**

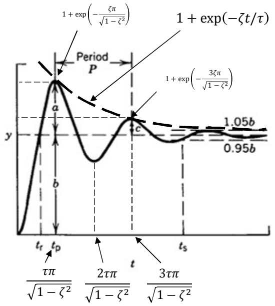
  $t_p = \tau\pi/\sqrt{1-\zeta^2}$

  – **Settling time (t_s)**

  $t_s \approx -\tau/\zeta \ln(0.05)$

  – **Overshoot (OS)**

  $OS = a/b = \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$



  – **Decay ratio (DR): a function of damping coefficient only!**

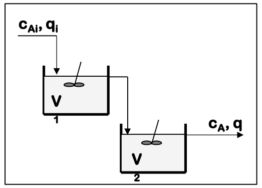
  $DR = c/a = (OS)^2 = \exp\left(-2\pi\zeta/\sqrt{1-\zeta^2}\right)$

  – **Period of oscillation (P)** $P = 2\pi\tau/\sqrt{1-\zeta^2}$

2^{ND} ORDER PROCESSES

- **Two tanks in series**
  - **If v_1=v_2, critically damped.**
  - **Or, overdamped (no oscillation)**

$$\frac{C_A(s)}{C_{Ai}(s)} == \frac{1}{((V_1/q)s+1)((V_2/q)s+1)}$$



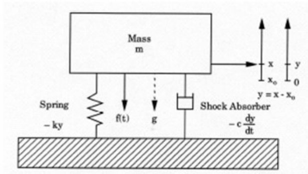
- **Spring-dashpot (shock absorber)**
  - **By force balance**

$$(mg+f(t)) - ky - cv = ma$$

$$my'' = -ky - cy' + (mg + f(t))$$

$$\Big(\sqrt{\frac{m}{k}}\Big)^2 y'' + 2\sqrt{\frac{c^2}{4mk}}\sqrt{\frac{m}{k}}\,y' + y = \tilde{f}(t)$$

τ $\quad$ ζ **(can be <1: underdamped)**



Underdamped Processes

- **Many examples can be found in mechanical and electrical system.**
- **Among chemical processes, open-loop underdamped process is quite rare.**
- **However, when the processes are controlled, the responses are usually underdamped.**
- **Depending on the controller tuning, the shape of response will be decided.**
- **Slight overshoot results short rise time and often more desirable.**
- **Excessive overshoot may result long-lasting oscillation.**

POLES AND ZEROS

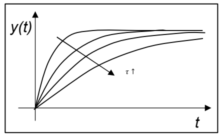
$$G(s) = \frac{N(s)}{D(s)} = \frac{K(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + 1)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1)}$$

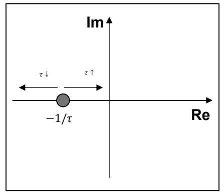
- **Poles ($D(s)$=0)**
  - **Where a transfer function cannot be defined.**
  - **Roots of the denominator of the transfer function**
  - **Modes of the response**
  - **Decide the stability**
- **Zero ($N(s)$=0)**
  - **Where a transfer function becomes zero.**
  - **Roots of the numerator of the transfer function**
  - **Decide weightings for each mode of response**
  - **Decide the size of overshoot or inverse response**
- **They can be real or complex**

- **Real pole from** $(\tau s + 1)$

$$s = -\frac{1}{\tau}$$

  - **Mode:** $e^{-t/\tau}$





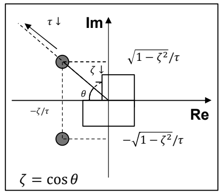
  - **If the pole is at the origin, it becomes "integrating pole."**
  - **If the pole is in RHP, the response increases exponentially.**

- **Complex pole from** $(\tau^2 s^2 + 2\zeta\tau s + 1)$ $(-1 < \zeta < 1)$

$$s = -\frac{\zeta}{\tau} \pm j\frac{\sqrt{1-\zeta^2}}{\tau} = -\alpha \pm j\beta$$

$$|s| = \sqrt{\frac{\zeta^2 + 1 - \zeta^2}{\tau^2}} = \frac{1}{\tau} \quad \text{(function of } \tau \text{ only)}$$

$$\measuredangle s = \pm\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \quad \text{(function of } \zeta \text{ only)}$$



– **Modes:**

$$e^{-\alpha t \pm j\beta t} = e^{-\alpha t}(\cos\beta\, t \pm j \sin\beta\, t)$$
$$= e^{-\zeta/\tau}\left(\cos\frac{\sqrt{1-\zeta^2}}{\tau}t \pm j\sin\frac{\sqrt{1-\zeta^2}}{\tau}t\right)$$

– **Assume τ is positive.**

– **If $\zeta < 0$, the exponential part will grow as t increases: unstable**

– **If $\zeta > 0$, the exponential part will shrink as t increases: stable**

– **If $\zeta = 0$, the roots are pure imaginary: sustained oscillation**

- **Effect of zero**

$$G(s) = \frac{N(s)}{(s+p_1)\cdots(s+p_n)} = w_1\frac{1}{(s+p_1)} + \cdots + w_n\frac{1}{(s+p_n)}$$

– **The effects on weighting factors are not obvious, but it is clear that the numerator (zeros) will change the weighting factors.**

EFFECTS OF ZEROS

- **Lead-lag module**

$$G(s) = \frac{N(s)}{D(s)} = \frac{K(\tau_a s+1)}{(\tau_1 s+1)} \begin{matrix}\longrightarrow \textbf{Lead}\\ \longrightarrow \textbf{Lag}\end{matrix}$$

– **Depending on the location of zero**

$$Y(s) = \frac{KM(\tau_a s+1)}{s(\tau_1 s+1)} = KM\left\{\frac{1}{s} + \frac{\tau_a - \tau_1}{\tau_1 s+1}\right\} \qquad y(t) = KM\left[1 - \left(1 - \frac{\tau_a}{\tau_1}\right)e^{-t/\tau_1}\right]$$

(a) $\tau_a > \tau_1 > 0$

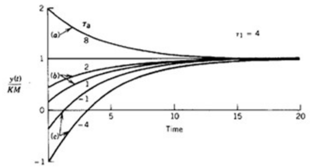
The lead dominates the lag.

(b) $0 \le \tau_a < \tau_1$

The lag dominates the lead.

(c) $0 > \tau_a$

Inverse response



- **Overdamped 2nd order+single zero system**

$$G(s) = \frac{N(s)}{D(s)} = \frac{K(\tau_a s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$Y(s) = \frac{KM(\tau_a s+1)}{s(\tau_1 s+1)(\tau_2 s+1)} = KM\left\{\frac{1}{s} + \frac{\tau_1(\tau_a-\tau_1)}{\tau_1-\tau_2}\frac{1}{\tau_1 s+1} + \frac{\tau_2(\tau_a-\tau_2)}{\tau_2-\tau_1}\frac{1}{\tau_2 s+1}\right\}$$

$$y(t) = KM\left[1 + \frac{\tau_a-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} + \frac{\tau_a-\tau_2}{\tau_2-\tau_1}e^{-t/\tau_2}\right]$$

(a) $\tau_a > \tau_1 > 0$ (assume $\tau_1 > \tau_2$)

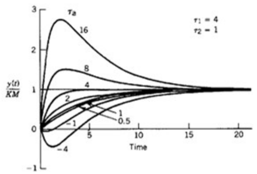
The lead dominates the lags.

(b) $0 < \tau_a \le \tau_1$

The lags dominate the lead.

(c) $0 > \tau_a$

Inverse response

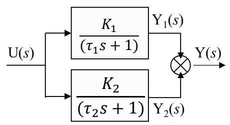


- **Other interpretation**

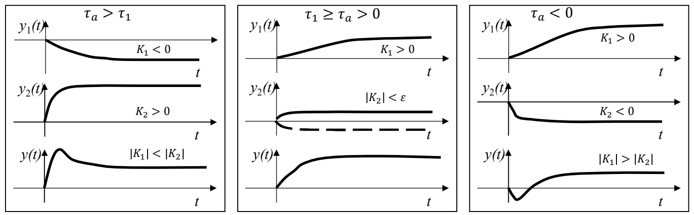
$$G(s) = \frac{K(\tau_a s+1)}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{K_1}{(\tau_1 s+1)} + \frac{K_2}{(\tau_2 s+1)}$$

$$K_1 = \left.\frac{K(\tau_a s+1)}{(\tau_2 s+1)}\right|_{s=-1/\tau_1} = \frac{K(\tau_1-\tau_a)}{(\tau_1-\tau_2)}$$

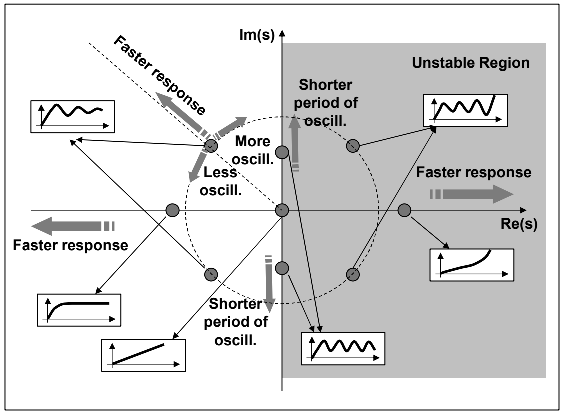
$$K_2 = \left.\frac{K(\tau_a s+1)}{(\tau_1 s+1)}\right|_{s=-1/\tau_2} = \frac{K(\tau_a-\tau_2)}{(\tau_1-\tau_2)}$$



– **Since $\tau_1 > \tau_2$, 1 is slow dynamics and 2 is fast dynamics.**



EFFECTS OF POLE LOCATION



EFFECTS OF ZERO LOCATION