Chapter 1

Mathematical Preliminaries

1.1 Notations

$\mathbf{R}^n$: the set of all n-dimensional real valued column vectors v

$(\mathbf{R}^n)^*$: the set of all n-dimensional real valued row vectors w^* that is called covector.

An element $w^* \in (\mathbf{R}^n)^*$ defines a linear map from $\mathbf{R}^n$ to $\mathbf{R}$ through inner product:

$$\langle w^*, v\rangle := w^* v = [w_1 \ \cdots \ w_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} w_i v_i$$

Smooth Vector Field: smooth mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$ = n-tuples of smooth functions stacked in column

Smooth Covector Field: smooth mappings from $\mathbf{R}^n$ to $(\mathbf{R}^n)^*$ = n-tuples of smooth functions stacked in row

Let λ be a mapping from $\mathbf{R}^n$ to $\mathbf{R}$ (a scalar field)

Gradient or Differential ($d\lambda$): a covector field of the form

$$d\lambda := \frac{d\lambda}{dx} = \left[\frac{\partial\lambda}{\partial x_1} \ \frac{\partial\lambda}{\partial x_2} \ \cdots \ \frac{\partial\lambda}{\partial x_n}\right]$$

Jacobian of a vector field f:

$$\frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Lie derivative of a scalar field λ along a vector field f: directional derivative of λ in the direction of f

$$\boldsymbol{L}_f \lambda := \langle d\lambda(x), f(x) \rangle = \frac{d\lambda}{dx} f(x) = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x)$$

Lie derivative is a scalar field.

Exs:

$$f(x) = \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1(x) = x_2^2 \Rightarrow \boldsymbol{L}_f \lambda_1 = 0$
$\lambda_2(x) = x_1^2 \Rightarrow \boldsymbol{L}_f \lambda_2 = 2x_1 x_2$
Notations:

$$\boldsymbol{L}_g \boldsymbol{L}_f \lambda = \frac{d(\boldsymbol{L}_f \lambda)}{dx} g(x)$$

$$\boldsymbol{L}_f^k \lambda := \underbrace{\boldsymbol{L}_f \cdots \boldsymbol{L}_f}_{k} \lambda$$

Lie bracket (product): another vector field

$$[f, g](x) = ad_f g := \boldsymbol{L}_f g - \boldsymbol{L}_g f = \frac{dg}{dx} f(x) - \frac{df}{dx} g(x)$$

Exs:
i) f, g constant $\Rightarrow [f, g] = 0$
ii) $[f, g] = 0 \Rightarrow \boldsymbol{L}_f g = \boldsymbol{L}_g f \Rightarrow f, g$ commute
iii) f, g linear vector fields, $f = Fx$, $g = Gx$

$$[f, g](x) = \boldsymbol{L}_f g - \boldsymbol{L}_g f = GFx - FGx = (GF - FG)x$$

($= 0$ if F, G commute as matrices)

Notation:

$$ad_f^k g(x) := \underbrace{[f,[f,\cdots,[f,g]]]}_{k} = [f, ad_f^{k-1} g](x)$$

Proposition: Lie bracket

1. is bilinear

$$[r_1 f_1 + r_2 f_2, g_1] = r_1[f_1, g_1] + r_2[f_2, g_1]$$
$$[f_1, r_1 g_1 + r_2 g_2] = r_1[f_1, g_1] + r_2[f_1, g_2]$$

2. is skew commutative

$$[f, g] = -[g, f]$$

3. satisfies Jacobi identity:

$$[f,[g,p]] + [g,[p,f]] + [p,[f,g]] = 0$$

Proof: Exercises

<u>An Interpretation of Lie Bracket</u>

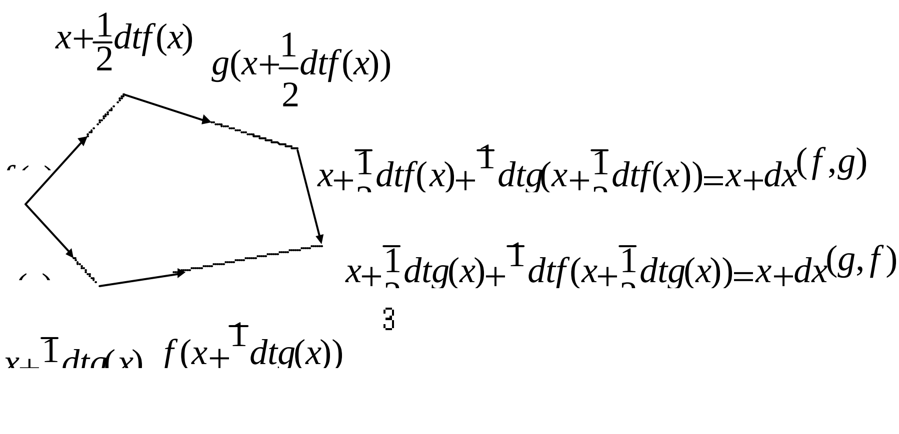
$$\dot{x} = h(x)$$

Case I:

$$h(x) = \begin{cases} f(x) & \text{for } t \in [0, \frac{1}{2}dt] \\ g(x) & \text{for } t \in [\frac{1}{2}dt, dt] \end{cases}$$

Case II:

$$h(x) = \begin{cases} g(x) & \text{for } t \in [0, \frac{1}{2}dt] \\ f(x) & \text{for } t \in [\frac{1}{2}dt, dt] \end{cases}$$



Take the difference:

$$x + dx^{(f,g)} - (x + dx^{(g,f)}) = dx^{(f,g)} - dx^{(g,f)}$$

$$= \frac{1}{2}dt \left[f(x) + g(x + \frac{1}{2}dt f(x)) - g(x) - f(x + \frac{1}{2}dt g(x)) \right]$$

$$\approx \frac{1}{2}dt \left[f(x) + g(x) + \frac{dg}{dx}(x) f(x) \frac{1}{2}dt - g(x) - f(x) - \frac{df}{dx}(x) g(x) \frac{1}{2}dt \right]$$

$$= \frac{1}{4}dt^2 \left[\frac{dg}{dx} f - \frac{df}{dx} g \right] = \frac{1}{4}dt^2 [f, g]$$

Derivative of a covector field ω along a vector field f: another covector field

$$\boldsymbol{L}_f \omega(x) = f^T(x) \left(\frac{d\omega^T}{dx} \right)^T + \omega(x) \frac{df}{dx}$$

Proposition: α, β, λ: real valued functions

1.
$$\boldsymbol{L}_{\alpha f} \lambda(x) = (\boldsymbol{L}_f \lambda(x)) \alpha(x)$$

2.
$$[\alpha f, \beta g](x) = \alpha(x)\beta(x)[f, g](x) + (\boldsymbol{L}_f \beta(x))\alpha(x) g(x) - (\boldsymbol{L}_g \alpha(x))\beta(x) f(x)$$

3.
$$\boldsymbol{L}_{[f,g]} \lambda(x) = \boldsymbol{L}_f \boldsymbol{L}_g \lambda(x) - \boldsymbol{L}_g \boldsymbol{L}_f \lambda(x)$$

4.
$$\boldsymbol{L}_{\alpha f} \beta w(x) = \alpha(x)\beta(x)(\boldsymbol{L}_f w(x)) + \beta(x)\langle w(x), f(x)\rangle d\alpha(x) + (\boldsymbol{L}_f \beta(x))\alpha(x) w(x)$$

5.
$$\boldsymbol{L}_f d\lambda(x) = d\boldsymbol{L}_f \lambda(x)$$

6.
$$\boldsymbol{L}_f \langle w, g\rangle(x) = \langle \boldsymbol{L}_f w(x), g(x)\rangle + \langle w(x), [f, g](x)\rangle$$

Proof: 1)

$$L_{\alpha f}\lambda(x) = \frac{d\lambda}{dx}\alpha(x)f(x) = \frac{d\lambda}{dx}f(x)\alpha(x) = (L_f\lambda(x))\alpha(x)$$

Other proofs are left as exercises.

Linear change of coordinates is defined by a linear mapping

$$z = Tx$$

where T is a nonsingular $n \times n$ matrix.

Nonlinear global (local) change of coordinates is defined by a vector field

$$z = \Phi(x)$$

that is a global (local) diffeomorphism:

i) $\Phi(x)$ is globally (locally) invertible, i.e. there exists a function $\Phi^{-1}(z)$ such that

$$\Phi^{-1}(\Phi(x)) = x \quad \forall x \in \mathbf{R}^n(U \subset \mathbf{R}^n)$$

ii) $\Phi(x)$ and $\Phi^{-1}(z)$ are both smooth mappings on $\mathbf{R}^n(U \subset \mathbf{R}^n)$.

Proposition: Suppose $\Phi(x)$ is a smooth vector field defined on some $U \subset \mathbf{R}^n$. Suppose the Jacobian matrix of Φ is nonsingular at a point $x = x^o$. Then, on a suitable open set $U^o \subset U$, containing x^o, $\Phi(x)$ defines a local diffeomorphism.

Ex:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Phi(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ \sin x_2 \end{bmatrix} \quad \forall x \in \mathbf{R}^2$$

Jacobian:

$$\frac{d\Phi}{dx} = \begin{bmatrix} 1 & 1 \\ 0 & \cos x_2 \end{bmatrix}$$

has rank 2 at $x^o = (0, 0)$. Indeed, Φ defines diffeomorphism on

$$U^o = \left\{(x_1, x_2) : |x_2| < \frac{\pi}{2}\right\}$$

For any set $V \supset U^o$, Φ is not diffeomorphism because it is not injective (one-to-one).

1.2 Distributions

Suppose $f_1, \cdots, f_d$ are (smooth) vector fields defined on $U \subset \mathbf{R}^n$.

Define (smooth) distribution:

$$\triangle(x) := span\{f_1(x), \cdots, f_d(x)\}$$

$= \{f(x) = a_1(x)f_1(x) + \cdots + a_d(x)f_d(x),\ a_1(x), \cdots, a_d(x)$ smooth scalar fields$\}$.

Given $x \in U$, $\triangle(x)$ is a subspace in $\mathbf{R}^n$.

Notations:

1. $(\triangle_1 + \triangle_2)(x) = \triangle_1(x) + \triangle_2(x)$
2. $(\triangle_1 \cap \triangle_2)(x) = \triangle_1(x) \cap \triangle_2(x)$
3. $\triangle_1 \supset \triangle_2$ if $\triangle_1(x) \supset \triangle_2(x)$ for all x.
4. $f \in \triangle$ if $f(x) \in \triangle(x)$ for all x.

Suppose F is an $n \times n$ matrix whose columns are smooth vector fields.

Then the distribution defined by n columns of F is the image space of the matrix F:

$$\triangle(x) = Im(F(x)).$$

Dim$\triangle(x)$ = rank of $F(x)$

Ex: $U = \mathbf{R}^3$

$$F(x) = \begin{bmatrix} x_1 & x_1x_2 & x_1 \\ 1 + x_3 & (1 + x_3)x_2 & x_1 \\ 1 & x_2 & 0 \end{bmatrix}$$

$$\triangle(x) = span\left\{\begin{bmatrix} 0 \\ 1 + x_3 \\ 1 \end{bmatrix}\right\} \quad \text{if } x_1 = 0$$

$$\triangle(x) = span\left\{\begin{bmatrix} x_1 \\ 1 + x_3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \quad \text{if } x_1 \neq 0$$

If $\triangle_1, \triangle_2$ are smooth, $\triangle_1 + \triangle_2$ is smooth.

However, $\triangle_1 \cap \triangle_2$ may not be smooth although $\triangle_1, \triangle_2$ are smooth.

Ex:

$$\triangle_1(x) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \qquad \triangle_2(x) = span\left\{\begin{bmatrix} 1+x_1 \\ 1 \end{bmatrix}\right\}$$

$$(\triangle_1 \cap \triangle_2)(x) = \begin{cases} \{0\} & \text{if } x_1 \neq 0 \\ \triangle_1(x) = \triangle_2(x) & \text{if } x_1 = 0 \end{cases}$$

Definition: $\triangle$ is nonsingular if $\exists\, d$ such that

$$\dim(\triangle(x)) = d \quad \forall x \in U$$

A singular distribution is also called a distribution of variable dimension.

A point $x^o \in U$ is regular if there exists a nbhd U^o of x^o on which $\triangle$ is nonsingular.

A nonregular point $x^o \in U$ is called singular.

Ex:

$$F(x) = \begin{bmatrix} x_1 & x_1 x_2 & x_1 \\ 1+x_3 & (1+x_3)x_2 & x_1 \\ 1 & x_2 & 0 \end{bmatrix}$$

$$\dim(\triangle(x)) = \begin{cases} 2 & \text{if } x_1 \neq 0 \\ 1 & \text{if } x_1 = 0 \end{cases}$$

The set of singular points is $\{x \in \mathbf{R}^3 : x_1 = 0\}$.

Lemma: Let $\triangle$ be a smooth distribution and x^o a regular point of $\triangle$. Suppose $\dim(\triangle(x^o)) = d$. Then there exist an open neighborhood U^o of x^o and a set $\{f_1, \cdots, f_d\}$ of smooth vector fields defined on U^o with the property that

i) the vectors $f_1(x), \cdots, f_d(x)$ are linearly independent at each $x \in U^o$

ii) $\triangle(x) = span\{f_1(x), \cdots, f_d(x)\}$ at each $x \in U^o$.

Moreover, every smooth vector field $\tau \in \triangle$ can be expressed, on U^o, as

$$\tau(x) = \sum_{i=1}^{d} c_i(x) f_i(x)$$

where $c_1(x), \cdots, c_d(x)$ are smooth scalar fields defined on U^o.

Lemma: The set of all regular points of a distribution $\triangle$, defined on U, is an open and dense subset of U.

Lemma: Let $\triangle_1$ and $\triangle_2$ be two smooth distributions, defined on $\boldsymbol{U}$, with the property that $\triangle_2$ is nonsingular and $\triangle_1(x) \subset \triangle_2(x)$ at each points x of a dense subset of $\boldsymbol{U}$. Then $\triangle_1 \subset \triangle_2$.

Lemma: Let $\triangle_1$ and $\triangle_2$ be two smooth distributions, defined on $\boldsymbol{U}$, with the property that $\triangle_1$ is nonsingular, $\triangle_1 \subset \triangle_2$ and $\triangle_1(x) = \triangle_2(x)$ at each points x of a dense subset of $\boldsymbol{U}$. Then $\triangle_1 = \triangle_2$.

Lemma: Let x^o be a regular point of $\triangle_1, \triangle_2$, and $\triangle_1 \cap \triangle_2$. Then there exists a nbhd $\boldsymbol{U}^o$ of x^o such that the restriction of $\triangle_1 \cap \triangle_2$ to $\boldsymbol{U}^o$ is smooth.

Def: $\triangle$ is involutive if

$$\tau_1 \in \triangle, \tau_2 \in \triangle \quad \Rightarrow \quad [\tau_1, \tau_2] \in \triangle.$$

Fact: $\triangle$: nonsingular, $\tau_1, \tau_2 \in \triangle \Rightarrow$

$$\tau_1(x) = \sum_{i=1}^{d} c_i(x) f_i(x) \qquad \tau_2(x) = \sum_{i=1}^{d} d_i(x) f_i(x).$$

Then TFAE

a) $\triangle$ is involutive

b) $[f_i, f_j] \in \triangle \quad \forall 1 \leq i, j \leq d$

c) $rank(f_1(x) \ \cdots \ f_d(x)) = rank(f_1(x) \ \cdots \ f_d(x) \ [f_i, f_j](x)) \quad \forall x \ \ \forall 1 \leq i, j \leq d$

Proof: (a $\Rightarrow$ b) Obvious

(a $\Leftarrow$ b)

$$[\tau_1, \tau_2] = \left[\sum_{i=1}^{d} c_i f_i, \sum_{j=1}^{d} d_j f_j\right] = \sum_{i=1}^{d}\sum_{j=1}^{d} [c_i f_i, d_j f_j]$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\left(\boldsymbol{L}_{c_i f_i} d_j f_j - \boldsymbol{L}_{d_j f_j} c_i f_i\right) = \sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{dd_j f_j}{dx} c_i f_i - \frac{dc_i f_i}{dx} d_j f_j\right)$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\left[\left(f_j dd_j + d_j \frac{df_j}{dx}\right) c_i f_i - \left(f_i dc_i + c_i \frac{df_i}{dx}\right) d_j f_j\right]$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\left(c_i d_j [f_i, f_j] + c_i (\boldsymbol{L}_{f_i} d_j) f_j - d_j (\boldsymbol{L}_{f_j} c_i) f_i\right)$$

(a $\Leftrightarrow$ c) Obvious.

Ex1:

$$\triangle = span\{f_1, f_2\} \qquad (**)$$

where

$$f_1(x) = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix} \qquad f_2(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}$$

$\dim \triangle = 2$ for all $x \in \mathbf{R}^3$

$$[f_1, f_2](x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$rank(f_1 \ f_2 \ [f_1, f_2])(x) = rank \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3$$

$\Rightarrow$ $\triangle$ is not involutive.

Ex 2: $\boldsymbol{U} = \{x \in \mathbf{R}^3 : x_1^2 + x_3^2 \neq 0\} = \mathbf{R}^3 \setminus \{x \in \mathbf{R}^3 : x = (0, x_2, 0)\}$

$$\triangle = span\{f_1, f_2\}$$

where

$$f_1(x) = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix} \qquad f_2(x) = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}$$

$\dim \triangle = 2$ for all $x \in \boldsymbol{U}$

$$[f_1, f_2](x) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$rank(f_1 \ f_2 \ [f_1, f_2])(x) = rank \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2$$

$\Rightarrow$ $\triangle$ is involutive.

Fact: Any one dimensional distribution is involutive ($f_1 = f$ and $[f_1, f_1] = 0$).

Fact: $\triangle_1, \triangle_2$: involutive $\Rightarrow$ $\triangle_1 \cap \triangle_2$: involutive

However, $\triangle_1 + \triangle_2$ may not be involutive even if $\triangle_1, \triangle_2$ are involutive

Ex: Consider (**). Let $\triangle = \triangle_1 + \triangle_2$ where

$$\triangle_1 := span\{f_1\} \quad \triangle_2 := span\{f_2\}$$

$\triangle_1, \triangle_2$: involutive

$\triangle$: not involutive

Def: codistribution:

$$\Omega(x) := span\{w_1(x), \cdots, w_d(x)\}$$

Given $x \in \boldsymbol{U}$, $\Omega(x)$ is a subspace of $(\mathbf{R}^n)^*$.

Def: Annihilator ($\triangle^\perp(x)$) of $\triangle(x)$:

$$\triangle^\perp(x) := \{w^* \in (\mathbf{R}^n)^* : \langle w^*, v\rangle = 0 \; \forall v \in \triangle(x)\}$$

a codistribution.

Def: Annihilator ($\Omega^\perp(x)$) of $\Omega(x)$:

$$\Omega^\perp(x) := \{v \in \mathbf{R}^n : \langle w^*, v\rangle = 0 \; \forall w^* \in \Omega(x)\}$$

a distribution.

Fact: the annihilator of a smooth distribution may not be smooth.

Ex:

$$\triangle = span\{x\}$$

$$\Downarrow$$

$$\triangle^\perp(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ \mathbf{R}^* & \text{if } x = 0 \end{cases}$$

$\Rightarrow$ not smooth.

Fact: the annihilator of a nonsmooth distribution may be smooth.

Ex:

$$\triangle_1(x) = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \qquad \triangle_2(x) = span\left\{\begin{bmatrix} 1 + x_1 \\ 1 \end{bmatrix}\right\}$$

$\triangle_1 \cap \triangle_2$: not smooth

$$[\triangle_1 \cap \triangle_2]^\perp(x) = \begin{cases} (\mathbf{R}^2)^* & \text{if } x \neq 0 \\ span\{(1 \ -1)\} & \text{if } x = 0 \end{cases}$$

smooth because it is spanned by

$$w_1 = (1 \ -1)$$

$$w_2 = (1 \ -(1 - x_1))$$

Properties:

1. $\dim \triangle + \dim \triangle^\perp = n$
2. $\triangle_1 \supset \triangle_2$ iff $\triangle_1^\perp \subset \triangle_2^\perp$
3. $[\triangle_1 \cap \triangle_2]^\perp = \triangle_1^\perp + \triangle_2^\perp$

Suppose $\triangle = span\{\text{columns of } \boldsymbol{F}\}$. Then

$$\triangle^\perp(x) = \{w^* \in (\mathbf{R}^n)^* : w^* \boldsymbol{F}(x) = 0\}$$

Suppose $\Omega = span\{\text{rows of } \boldsymbol{W}\}$. Then

$$\Omega^\perp(x) = \{v \in \mathbf{R}^n : \boldsymbol{W}(x)v = 0\} = ker(\boldsymbol{W}(x))$$

The similar lemmata to distribution hold for codistribution.

Lemma: Let x^o be a regular point of smooth $\triangle$. Then x^o is a regular point of $\triangle^\perp$ and $\exists$ nbhd $\boldsymbol{U}^o$ of x^o such that the restriction of $\triangle^\perp$ to $\boldsymbol{U}^o$ is a smooth codistribution.

$\triangle$ nonsingular distribution $\Rightarrow$

$$\triangle(x) = span\{f_1(x) \cdots f_d(x)\}$$

where $f_1, \cdots, f_d$ are smooth vector fields.

$\triangle^\perp$: smooth, nonsingular, $\dim \triangle^\perp = n - d \Rightarrow \triangle^\perp = span\{w_1(x) \cdots w_{n-d}(x)\}$
Then

$$\langle w_j(x), f_i(x) \rangle = 0 \quad \forall 1 \leq i \leq d, \ 1 \leq j \leq n - d, \ \forall x \in \boldsymbol{U}^o$$

or

$$w_j(x)\boldsymbol{F}(x) = 0 \quad (*)$$

where

$$F(x) = [f_1(x) \ \cdots \ f_d(x)].$$

Suppose one seeks the solution of (*) in the form of exact differential:

$$w_j = \frac{d\lambda_j}{dx}.$$

Namely, find $n-d$ linearly independent solutions of

$$\frac{d\lambda_j}{dx}(f_1(x) \ \cdots \ f_d(x)) = \frac{d\lambda_j}{dx}F(x) = 0.$$

Question: When does there exist the solution to this problem?

Def: A nonsingular Δ is compeletely integrable if, for each $x^o \in U$, $\exists$ nbhd U^o and $n-d$ real valued functions $\lambda_1, \cdots \lambda_{n-d}$ defined on U^o such that

$$span\{d\lambda_1, \cdots, d\lambda_{n-d}\} = \Delta^{\perp} \quad on \ U^o.$$

Complete Integrability? $\Leftrightarrow$ Question

Frobenius Theorem: A nonsingular distribution is completely integrable iff it is involutive.

<u>An Application of Frobenius Theorem</u>

Let Δ be involutive and $d\lambda_1, \cdots, d\lambda_{n-d}$ be $n-d$ linearly independent solutions to Question at x^0.

Among $x_1(x) = x_1, \cdots, x_n(x) = x_n$, $\exists$ d functions whose differentials are linearly independent such that, together with the differentials of λ's, form n linearly independent covectors. Let $\phi_1, \cdots, \phi_d$ denotes the functions thus chosen and

$$\phi_{d+1}(x) = \lambda_1(x), \cdots, \phi_n = \lambda_{n-d}(x)$$

$\Rightarrow$ the Jacobian of

$$z = \Phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

is nonsingular at $x^0 \Rightarrow \Phi$ local diffeomorphism around x^0.

Suppose $\dot{x} = \tau \in \Delta \Rightarrow \tau = \sum_{i=1}^{d} c_i f_i$

$$\dot{z} = \left[\frac{d\Phi}{dx}\tau(x)\right]_{x=\Phi^{-1}(z)} =: \bar{\tau}(z).$$

The last $n-d$ rows of the Jacobian of Φ are $d\lambda_1, \cdots, d\lambda_{n-d}$ and span $\triangle^\perp$ $\Rightarrow$

$$\tau(z) = \begin{bmatrix} \tau_1(z) \\ \vdots \\ \tau_d(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For $1 \leq i \leq k$, let $\triangle_i$ be a distribution with dimension d_i such that

$$\triangle_1 \supset \cdots \supset \triangle_k.$$

$\triangle_1$ involutive around $x^0 \Rightarrow \exists\ \lambda_i$, $1 \leq i \leq n - d_1$ such that

$$span\{d\lambda_1, \cdots, d\lambda_{n-d_1}\} = \triangle_1^\perp.$$

$\triangle_2$ involutive $\Rightarrow \exists\ \mu_i$, $1 \leq i \leq n - d_2$ such that

$$span\{d\mu_1, \cdots, d\mu_{n-d_2}\} = \triangle_2^\perp.$$

$\triangle_1^\perp \subset \triangle_2^\perp \Rightarrow \exists\ \mu_{n-d_1+1}, \cdots, \mu_{n-d_2}$ such that

$$span\{d\lambda_1, \cdots, d\lambda_{n-d_1}\} + span\{d\mu_{n-d_1+1}, \cdots, d\mu_{n-d_2}\} = \triangle_2^\perp.$$

Corollary: Let $\triangle_1 \supset \cdots \supset \triangle_k$. $\triangle_k$'s are involutive iff $\exists$ a nbhd U^0 of x^0 and

$$\lambda_1^1, \cdots, \lambda_{n-d_1}^1, \lambda_1^2, \cdots, \lambda_{d_1-d_2}^2, \cdots, \lambda_1^k, \cdots, \lambda_{d_{k-1}-d_k}^k$$

defined on U^0 such that

$$\triangle_1^\perp = span\{d\lambda_1^1, \cdots, d\lambda_{n-d_1}^1\}$$

$$\triangle_i^\perp = \triangle_{i-1}^\perp + span\{d\lambda_1^i, \cdots, d\lambda_{d_{i-1}-d_i}^i\}$$

for $2 \leq i \leq k$.

Invariant Distributions

Notation: $[f, \triangle] := span\{[f, \tau],\ \tau \in \triangle\}$

$\triangle$ is invariant under a vector field f if

$$\tau \in \triangle \ \Rightarrow\ [f, \tau] \in \triangle$$

$$\Updownarrow$$
$$[f, \triangle] \subset \triangle.$$

Fact: $\triangle$ nonsingular: $\tau \in \triangle \Rightarrow$

$$\tau = \sum_{i=1}^{d} c_i(x)\tau_i(x)$$

$\Rightarrow$ $\triangle$ is invariant under f iff

$$[f, \tau_i] \in \triangle \quad \forall 1 \leq i \leq d.$$

Proof: $(\Rightarrow)$ Obvious
$(\Leftarrow)$

$$[f, \tau] = \frac{d\tau}{dx}f - \frac{df}{dx}\tau = \sum_{i=1}^{d}(\tau_i dc_i f + c_i \frac{d\tau_i}{dx}f - c_i \frac{df}{dx}\tau_i)$$
$$= \sum_{i=1}^{d} c_i[f, \tau_i] + \sum_{i=1}^{d}(L_f c_i)\tau_i.$$

From the proof of the above fact, notice that

$$[f, \triangle] \underset{\not\subset}{\supset} span\{[f, \tau_1], \cdots, [f, \tau_d]\}.$$

But

$$\triangle + [f, \triangle] = \triangle + span\{[f, \tau_1], \cdots, [f, \tau_d]\} = span\{\tau_1, \cdots, \tau_d, [f, \tau_1], \cdots, [f, \tau_d]\}.$$

<u>Connection between invariance of a distribution under a vector field and invariance of a subspace under a linear mapping</u>
V: subspace of $\mathbf{R}^n$ such that $AV \subset V$ (invariant under A)
Let $\triangle(x) = V$ and $f(x) = Ax$.
Let $v_1, \cdots, v_d$ be a basis for V and define

$$\tau_i(x) = v_i.$$

Then
$$[f, \tau_i] = \frac{d\tau_i}{dx}f - \frac{df}{dx}\tau_i = -Av_i \in V$$

and, thus, $\triangle$ is invariant under f.

Lemma (*): Let $\triangle$ be a nonsingular involutive distribution of dimension d and suppose $\triangle$ is invariant under f. Then at each x^0, $\exists$ a neighborhood U^0 of x^0 and a coordinate transformation $z = \Phi(x)$ defined on U^0, in which $\bar{f}(z) := \left[\frac{d\Phi}{dx} f(x)\right]_{x=\Phi^{-1}(z)}$ is represented by a vector of the form

$$\bar{f}(z) = \begin{bmatrix} \bar{f}_1(z_1, \cdots, z_d, z_{d+1}, \cdots, z_n) \\ \vdots \\ \bar{f}_d(z_1, \cdots, z_d, z_{d+1}, \cdots, z_n) \\ \bar{f}_{d+1}(z_{d+1}, \cdots, z_n) \\ \vdots \\ \bar{f}_n(z_{d+1}, \cdots, z_n) \end{bmatrix}.$$

Proof: Assumptions $\Rightarrow$ $\triangle$: integrable
$\Rightarrow$ $\exists$ a neighborhood U^0 and $z = \Phi(x)$ defined on U^0 such that

$$span\{d\phi_{d+1}, \cdots, d\phi_n\} = \triangle^{\perp}.$$

Suppose $\bar{\tau}_i = e_i$. Then

$$[\bar{f}, \bar{\tau}_i] = -\frac{d\bar{f}}{dz}\bar{\tau}_i = -\frac{d\bar{f}}{dz_i}.$$

For $i = 1, \cdots, d$, the last $n - d$ components of $\bar{\tau}_i$ are 0 in z coordinate and, thus, $\tau_i \in \triangle$.

$\triangle$ invariant under $f \Rightarrow$ For $i = 1, \cdots, d$, $[f, \tau_i] \in \triangle$ and, thus, the last $n - d$ components of $\left[\frac{d\Phi}{dx}[f, \tau_i]\right]_{x=\Phi^{-1}(z)} = [\bar{f}, \bar{\tau}_i]$ are 0.

$$\Downarrow$$

$$\frac{d\bar{f}_k}{dz_i} = 0 \quad d+1 \le k \le n,\ 1 \le i \le d.$$

Lemma: Let $\triangle$ be a distribution invariant under f_1 and f_2. Then $\triangle$ is also invariant under $[f_1, f_2]$.

The proof of this lemma is straight forward in view of Jacobi Identity.

Remark: Let $smt(\triangle)$ be the largest smooth distribution contained in $\triangle$.

For nonsmooth $\triangle$, $\triangle$ is invariant under f if

$$[f, smt(\triangle)] \subset \triangle.$$

Since $[f, smt(\triangle)]$ is smooth,

$$[f, smt(\triangle)] \subset smt(\triangle).$$

Ω is invariant under f if

$$\omega \in \Omega \;\Rightarrow\; \boldsymbol{L}_f\omega \in \Omega$$

$$\Updownarrow$$

$$\boldsymbol{L}_f\Omega = span\{\boldsymbol{L}_f\omega : \omega \in \Omega\} \subset \Omega.$$

Lemma: If $\triangle$ is invariant under f, $\Omega = \triangle^\perp$ is also invariant under f. If Ω is invariant under f, $\triangle = \Omega^\perp$ is also invariant under f.

Chapter 2

Controllability and Observability

Def: A system is (locally) controllable if, for all $x_0, x_1 \in \mathbf{R}^n (U \subset \mathbf{R}^n)$, there exists input u that steers x_0 to x_1 in finite time.

Def: A system is (locally) controllable to x_1 if, for all $x_0 \in \mathbf{R}^n (U \subset \mathbf{R}^n)$, there exists input u that steers x_0 to x_1 in finite time.

Def: A system is (locally) controllable from x_0 if, for all $x_1 \in \mathbf{R}^n (U \subset \mathbf{R}^n)$, there exists input u that steers x_0 to x_1 in finite time.

Remark: Controllability from the origin is also called Reachability.

Remark: For "continuous" time linear systems,

controllability $\Leftrightarrow$ controllability to the origin $\Leftrightarrow$ reachability

Def: A system is observable if, for all input u and the corresponding output y, the initial state x is uniquely determined.

Linear in Input Systems:

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

$$y = h(x)$$

2.1 Linear Systems

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Assumptions:

i) $\exists$ d-dimensional subspace V of $\mathbf{R}^n$ such that V is invariant under A ($Ax \in V, \ \forall x \in V$)

ii) $Bu \in V, \forall u \in \mathbf{R}^m$

iii) V smallest subspace that satisfies i) and ii)

$\exists$ T such that $z = Tx = [z_1 \ \cdots \ z_d \ 0 \ \cdots \ 0]$ for all $x \in V$

$\Rightarrow$ $TV = \mathbf{R}^d \times \{0\} \subset \mathbf{R}^n$

Assumption i) $\Rightarrow$ $TAT^{-1}z \in TV, \ \forall z \in TV \Rightarrow$

$$TAT^{-1} =: \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

Assumption ii) $\Rightarrow$ $TBu \in TV, \forall u \in \mathbf{R}^m \Rightarrow$

$$TB =: \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\dot{z}_1 = \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 + \tilde{B}_1 u$$
$$\dot{z}_2 = \tilde{A}_{22}z_2$$

$$\Downarrow$$

$$z_1(T) = e^{\tilde{A}_{11}T}z_1(0) + \int_0^T e^{\tilde{A}_{11}(T-\tau)}\tilde{A}_{12}e^{\tilde{A}_{22}\tau}d\tau z_2(0) + \int_0^T e^{\tilde{A}_{11}(T-\tau)}\tilde{B}_1 u(\tau)d\tau$$
$$z_2(T) = e^{\tilde{A}_{22}T}z_2(0)$$
$$\Downarrow$$

Controllable point at t_f from $z(0) = Tx(0)$ has the form:

$$z(t_f) = z^o(t_f) + v$$

where

$$z^o(t_f) := e^{\tilde{A}t_f}z(0) \quad \text{and} \quad v \in TV$$

From the linear system theory, it holds that

Assumption iii) $\Leftrightarrow$ $\boldsymbol{TV} = \boldsymbol{Im}(\tilde{\boldsymbol{B}}\ \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} \cdots\ \tilde{\boldsymbol{A}}^{n-1}\tilde{\boldsymbol{B}}) = \boldsymbol{Im}(\tilde{\boldsymbol{B}}_1\ \tilde{\boldsymbol{A}}_{11}\tilde{\boldsymbol{B}}_1\ \cdots\ \tilde{\boldsymbol{A}}_{11}^{d-1}\tilde{\boldsymbol{B}}_1) \times \{0\}$ (controllable subspace)

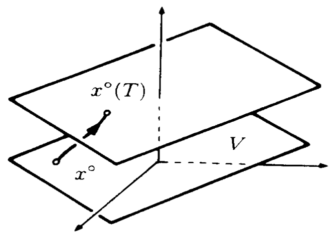
$$\Downarrow$$

$$rank(\tilde{\boldsymbol{B}}_1\ \tilde{\boldsymbol{A}}_{11}\tilde{\boldsymbol{B}}_1\ \cdots\ \tilde{\boldsymbol{A}}_{11}^{d-1}\tilde{\boldsymbol{B}}_1) = d$$

Hence, $(\tilde{\boldsymbol{A}}_{11}, \tilde{\boldsymbol{B}}_1)$ is controllable.

Under Assumptions i), ii) and iii), the set of all points controllable at t_f from $x(0)$ is

$$S_{e^{At_f}x(0)} := \{x \in \mathbf{R}^n : x = e^{At_f}x(0) + v,\ v \in \boldsymbol{V}\}$$



The system is controllable iff $\boldsymbol{V} = \mathbf{R}^n$ iff $\boldsymbol{V} = \boldsymbol{Im}(\boldsymbol{B}\ \boldsymbol{AB}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B})$ iff $rank[\boldsymbol{B}\ \boldsymbol{AB}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B}] = n$

Assumptions:

i) $\exists$ d-dimensional subspace $\boldsymbol{W}$ of $\mathbf{R}^n$ such that $\boldsymbol{W}$ is invariant under $\boldsymbol{A}$

ii) $Cx = 0, \forall x \in \boldsymbol{W}$

iii) $\boldsymbol{W}$ largest subspace that satisfies i) and ii)

i) and ii) $\Rightarrow$ $\exists$ $\boldsymbol{T}$ such that

$$\dot{z}_1 = \tilde{\boldsymbol{A}}_{11}z_1 + \tilde{\boldsymbol{A}}_{12}z_2 + \tilde{\boldsymbol{B}}_1 u$$

$$\dot{z}_2 = \tilde{\boldsymbol{A}}_{22}z_2 + \tilde{\boldsymbol{B}}_2 u$$

$$y = \tilde{C}_2 z_2$$

Assumption iii) $\Leftrightarrow$ $\boldsymbol{W} = ker \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \{0\} \times ker \begin{bmatrix} \tilde{C}_2 \\ \tilde{C}_2\tilde{A}_{22} \\ \vdots \\ \tilde{C}_2\tilde{A}_{22}^{n-d-1} \end{bmatrix}$ (unobservable subspace)

$$\Downarrow$$

$$rank \begin{bmatrix} \tilde{C}_2 \\ \tilde{C}_2\tilde{A}_{22} \\ \vdots \\ \tilde{C}_2\tilde{A}_{22}^{n-d-1} \end{bmatrix} = n - d$$

Hence, $(\tilde{C}_2, \tilde{A}_{22})$ is observable.

Under Assumptions i), ii) and iii), the set of all indistinguishable points at t_f is

$$S_{x(t_f)} := \{x \in \mathbf{R}^n : x = x(t_f) + w,\ w \in \boldsymbol{W}\}.$$

The system is observable iff $\boldsymbol{W} = \{0\}$ iff $ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \{0\}$ iff $rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$.

2.2 Local Controllability

Proposition: Let $\triangle$ be a nonsingular involutive distribution of dimension d and suppose $\triangle$ is invariant under f, g_1, $\cdots$, g_m. Moreover, suppose $span\{g_1, \cdots, g_m\} \subset \triangle$. Then for each x^0, $\exists$ a neighborhood $\boldsymbol{U}^0$ of x^0 and $z = \Phi(x)$ defined on $\boldsymbol{U}^0$ such that

$$\dot{\zeta}_1 = f_1(\zeta_1, \zeta_2) + \sum_{i=1}^{m} g_{1i}(\zeta_1, \zeta_2) u_i$$

$$\dot{\zeta}_2 = f_2(\zeta_2)$$
$$y = h(\zeta_1, \zeta_2)$$

where $\zeta_1 = (z_1, \cdots, z_d)$ and $\zeta_2 = (z_{d+1}, \cdots, z_n)$.

The proof is obvious from the "application of Frobenius Theorem" and Lemma (*).

For local decomposition in the above proposition, $\triangle$ must

i) be nonsingular

ii) be involutive

iii) contain the distribution $\bar{\triangle} = span\{g_1, \cdots, g_m\}$

iv) be invariant under $f, g_1, \cdots, g_m$

Question: Minimal such $\triangle$?

Lemma: Let $\bar{\triangle}$ be a smooth distribution and $\tau_1, \cdots, \tau_q$ vector fields. The family $\mathcal{F}$ of all distributions which are invariant under $\tau_1, \cdots, \tau_q$ and contain $\bar{\triangle}$ has a minimal element, which is a smooth distribution.

Proof: Clearly, $\mathcal{F} \neq \emptyset$ (Ex: $\triangle = \mathbf{R}^n$).

$\triangle_1, \triangle_2 \in \mathcal{F} \Rightarrow \triangle_1 \cap \triangle_2 \supset \bar{\triangle}$ and is invariant under $\tau_1, \cdots, \tau_q \Rightarrow \triangle_1 \cap \triangle_2 \in \mathcal{F}$

$\Rightarrow \hat{\triangle} := \cap_{\triangle \in \mathcal{F}} \triangle \supset \bar{\triangle}$ and is invariant under $\tau_1, \cdots, \tau_q \Rightarrow \hat{\triangle} \in \mathcal{F} \Rightarrow \hat{\triangle}$ is minimal

$\hat{\triangle}$ must be smooth because $\hat{\triangle} \supset smt(\hat{\triangle}) \supset \bar{\triangle}$ and $smt(\hat{\triangle})$ is invariant under $\tau_1, \cdots, \tau_q$.

$$\Downarrow$$

minimal $\triangle$ satisfying iii) and iv) always exists.

Notation: $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$: The minimal $\triangle$ satisfying iii) and iv).

Question: Is $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ involutive?

Lemma: Let

$$\triangle_0 = \bar{\triangle},$$
$$\triangle_k = \triangle_{k-1} + \sum_{i=1}^{q} [\tau_i, \triangle_{k-1}].$$

Then, for all k,

$$\triangle_k \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle.$$

If $\exists k^*$ such that $\triangle_{k^*} = \triangle_{k^*+1}$, then

$$\triangle_{k^*} = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle.$$

Proof: Suppose $\triangle_k \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$. Then

$$\triangle_{k+1} = \triangle_k + \sum_{i=1}^{q} [\tau_i, \triangle_k] = \triangle_k + \sum_{i=1}^{q} span\{[\tau_i, \tau] : \tau \in \triangle_k\}$$

$$\subset \triangle_k + \sum_{i=1}^{q} span\{[\tau_i, \tau] : \tau \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle\} \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle.$$

Hence, $\triangle_0 \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle \Rightarrow \triangle_k \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ for all k

Moreover, from the above equation, $\bar{\triangle} = \triangle_0 \subset \triangle_k$.

$\triangle_{k^*} = \triangle_{k^*+1} \Rightarrow \triangle_{k^*}$ is invariant under $\tau_1, \cdots, \tau_q$ because $[\tau_i, \triangle_{k^*}] \subset \triangle_{k^*+1} \subset \triangle_{k^*}$ for each i.

Question: When can the stopping condition be met?

$dim \triangle_k \leq dim \triangle_{k+1} \leq n$

Suppose $\triangle_k$'s are all nonsingular

$\Rightarrow dim \triangle_k < dim \triangle_{k+1} \leq n$ or $\triangle_k = \triangle_{k+1}$

$\Rightarrow k^* < n$

Suppose some $\triangle_k$'s are singular. Then we have the following lemma.

Lemma: $\exists$ an open and dense subset U^* of U such that, for each $x \in U^*$

$$\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x) = \triangle_{n-1}(x).$$

Proof: Suppose V open set on which $\triangle_{k^*}(x) = \triangle_{k^*+1}(x)$ for all $x \in V$.

The Proof of Previous Lemma $\Rightarrow \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x) \supset \triangle_{k^*}(x)$ for all $x \in V$.

Suppose $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x) \supsetneq \triangle_{k^*}(x)$ at some $x \in V$. Then define

$$\hat{\triangle}(x) := \begin{cases} \triangle_{k^*}(x) & \text{if } x \in V \\ \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x) & \text{if } x \notin V \end{cases}.$$

$\tau \in \hat{\triangle} \subset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle \Rightarrow [\tau_i, \tau] \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x)$

On V, $\tau \in \hat{\triangle} = \triangle_{k^*}$ and $[\tau_i, \triangle_{k^*}] \subset \triangle_{k^*+1} = \triangle_{k^*} \Rightarrow [\tau_i, \tau](x) \in \triangle_{k^*}(x)$, $\forall x \in V$.

$\Rightarrow \hat{\triangle} \supset \bar{\triangle}$ (because $= \triangle_0$) is invariant under $\tau_1, \cdots, \tau_q$ and is "smaller" than $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$.

$\Rightarrow$ contradiction $\Rightarrow \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle(x) = \triangle_{k^*}(x)$ for all $x \in V$.

U_k: Regular points of $\triangle_k \Rightarrow U_k$: open dense subset of U

$\Rightarrow U^* := \cap_{i=0}^{n-1} U_i$: open dense subset of U

In a nbhd of every $x \in U^*$, $\triangle_0, \cdots, \triangle_{n-1}$ are nonsingular.

$\Rightarrow\ \triangle_{n-1} = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ on $\boldsymbol{U}^*$.

Lemma: Suppose $\bar{\triangle}$ is the span of some vector fields in $\{\tau_1, \cdots, \tau_q\}$. Then $\exists$ an open and dense subset $\boldsymbol{U}^*$ of $\boldsymbol{U}$ with the following property. For each $x^0 \in \boldsymbol{U}^*$, $\exists$ a neighborhood $\boldsymbol{V}$ of x^0 and d vector fields (with $d = dim\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle (x^0)$) $\theta_1, \cdots, \theta_d$ of the form

$$\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]]$$

where $r \leq n-1$ (which may depend on i) and $v_0, \cdots, v_r$ are vector fields in the set $\{\tau_1, \cdots, \tau_q\}$ such that

$$\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle (x) = span\{\theta_1(x), \cdots, \theta_d(x)\}$$

for all $x \in \boldsymbol{V}$.

Proof: We will prove the lemma by induction.

$\boldsymbol{U}^*$ in the previous lemma. Let $d_0 = dim(\triangle_0) = dim(\bar{\triangle})$.

Assumption $\Rightarrow \triangle_0 = \bar{\triangle}$ is the span of some vector fields in $\{\tau_1, \cdots, \tau_q\}$.

$\Rightarrow\ \exists\ d_0$ vector fields in $\{\tau_1, \cdots, \tau_q\}$ that span $\triangle_0$.

Let $d_k = dim(\triangle_k)$ and suppose $\triangle_k$ is the span of d_k vector fields of

$$\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]]$$

where $v_0, \cdots, v_r$ are vector fields in $\{\tau_1, \cdots, \tau_q\}$.

$\tau \in \triangle_k \Rightarrow \tau = c_1\theta_1 + \cdots + c_{d_k}\theta_{d_k}$.

$$[\tau_j, c_1\theta_1 + \cdots + c_{d_k}\theta_{d_k}] = c_1[\tau_j, \theta_1] + \cdots + c_{d_k}[\tau_j, \theta_{d_k}] + (\boldsymbol{L}_{\tau_j} c_1)\theta_1 + \cdots + (\boldsymbol{L}_{\tau_j} c_{d_k})\theta_{d_k}$$

$$\Downarrow$$

$$\triangle_{k+1} = \triangle_k + [\tau_1, \triangle_k] + \cdots + [\tau_q, \triangle_k] = span\{\theta_i, [\tau_1, \theta_i], \cdots, [\tau_q, \theta_i] : 1 \leq i \leq d_k\}$$

$\triangle_{k+1}$ nonsingular $\Rightarrow\ \exists\ d_{k+1}$ vector fields of

$$\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]]$$

where $v_0, \cdots, v_r$ are vector fields in $\{\tau_1, \cdots, \tau_q\}$, which span $\triangle_{k+1}$ locally around x.

Lemma: Suppose $\bar{\triangle}$ is the span of some vector fields in $\{\tau_1, \cdots, \tau_q\}$ and $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ is nonsingular. Then $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ is involutive.

Proof: On U^*, $\sigma_1, \sigma_2 \in \triangle_{n-1} \Rightarrow$

$$\sigma_1 = \sum_{i=1}^{d} c_i^1 \theta_i, \qquad \sigma_2 = \sum_{i=1}^{d} c_i^2 \theta_i$$

where θ_i's are vector fields in the previous lemma. In a nbhd V of x,

$$[\sigma_1, \sigma_2] = \left[\sum_{i=1}^{d} c_i^1 \theta_i, \sum_{j=1}^{d} c_j^2 \theta_j\right] \in span\{\theta_i, [\theta_i, \theta_j] : 1 \le i, j \le d\}.$$

Previous lemma $\Rightarrow \triangle_{n-1}$ invariant under $\tau_1, \cdots, \tau_q$
$\Rightarrow \triangle_{n-1}$ invariant under $[\tau_i, \tau_j]$
By induction, $\triangle_{n-1}$ is invariant under $\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]]$
$\Rightarrow [\theta_i, \triangle_{n-1}] \subset \triangle_{n-1} \Rightarrow [\theta_i, \theta_j] \in \triangle_{n-1} \Rightarrow [\sigma_1, \sigma_2] \in \triangle_{n-1}$
$\Rightarrow$ on U^*, $[\sigma_1, \sigma_2] \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ for $\sigma_1, \sigma_2 \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$
Consider

$$\hat{\triangle} = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle + span\{[\theta_i, \theta_j] : \theta_i, \theta_j \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle\}$$

$$\Downarrow$$

$$\hat{\triangle} \supset \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$$

$$\Downarrow$$

$$\hat{\triangle}(x) = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle (x) \quad \text{on } U^*$$

$\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ nonsingular $\Rightarrow$ By the lemma on page 8, $\hat{\triangle} = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$.
$\Rightarrow [\theta_i, \theta_j] \in \hat{\triangle} = \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ for all $\theta_i, \theta_j \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$
$\Rightarrow [\sigma_1, \sigma_2] \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ for $\sigma_1, \sigma_2 \in \langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$

Lemma: Suppose $\bar{\triangle}$ is the span of some vector fields in $\{\tau_1, \cdots, \tau_q\}$ and $\triangle_{n-1}$ is nonsingular. Then $\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle$ is involutive and

$$\langle \tau_1, \cdots, \tau_q | \bar{\triangle} \rangle = \triangle_{n-1}.$$

Proof: The proof is obvious from the previous lemmata.

To recap, let $\tau_1 = f$, $\tau_{i+1} = g_i$, $i = 1, \cdots, m$, and $\bar{\triangle} = span\{\tau_2, \cdots, \tau_{m+1}\}$. Then
$\langle f, g_1, \cdots, g_m | span\{g_1, \cdots, g_m\} \rangle$ nonsingular
$\Rightarrow \langle f, g_1, \cdots, g_m | span\{g_1, \cdots, g_m\} \rangle$ is involutive, contain $span\{g_1, \cdots, g_m\}$ and is invariant under $\{f, g_1, \cdots, g_m\}$

$\Rightarrow$ Local decomposition is possible.

Suppose $\langle f, g_1, \cdots, g_m | span\{f, g_1, \cdots, g_m\}\rangle$ nonsingular

$$\Downarrow$$

Another local decomposition is possible with $\dot{\zeta}_2 = 0$.

Question: What is the relationship between these two decompositions?

Notation:

$$\boldsymbol{P} := \langle f, g_1, \cdots, g_m | span\{g_1, \cdots, g_m\}\rangle$$

$$\boldsymbol{R} := \langle f, g_1, \cdots, g_m | span\{f, g_1, \cdots, g_m\}\rangle$$

Lemma: $\boldsymbol{P}$ and $\boldsymbol{R}$ are such that

a) $\boldsymbol{P} + span\{f\} \subset \boldsymbol{R}$

b) if x is a regular point of $\boldsymbol{P}+span\{f\}$, then $(\boldsymbol{P}+span\{f\})(x) = \boldsymbol{R}(x)$.

Proof: $\boldsymbol{P} \subset \boldsymbol{R}$ and $f \in \boldsymbol{R} \Rightarrow$ a)

Previous lemma $\Rightarrow$ On $\boldsymbol{U}^*$, $\boldsymbol{R}$ is spanned by

$$\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]]$$

where $v_i \in \{f, g_1, \cdots, g_m\}$.

Either

i) $\theta_i = f \in span\{f\}$

or

ii) WLOG, $v_0 \in \{g_1, \cdots, g_m\}$ ($v_0 = v_1 = f \Rightarrow [v_1, v_0] = 0$ or $v_0 = f, v_1 = g_j \Rightarrow \theta_i = [v_r, [v_{r-1}, \cdots, [f, g_j]]]$ is OK)

For $v_i \in \{f, g_1, \cdots, g_m\}$, $g_j \in \boldsymbol{P}$ and $\boldsymbol{P}$ invariant under $f, g_1, \cdots, g_m \Rightarrow$

$$\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, g_j]]] \in \boldsymbol{P}.$$

i), ii) $\Rightarrow$ On $\boldsymbol{U}^*$, $\theta_i = [v_r, [v_{r-1}, \cdots, [v_1, v_0]]] \in \boldsymbol{P} + span\{f\}$

$\Rightarrow$ On $\boldsymbol{U}^*$, $\boldsymbol{R} \subset \boldsymbol{P} + span\{f\} \Rightarrow$ On $\boldsymbol{U}^*$, $\boldsymbol{R} = \boldsymbol{P} + span\{f\}$.

x regular point $\Rightarrow \boldsymbol{P} + span\{f\}$ nonsingular on a nbhd $\boldsymbol{V}$ of $x \Rightarrow$ On $\boldsymbol{V}$, $\boldsymbol{R}(x) = \boldsymbol{P} + span\{f\}(x)$.

Corollary: If $\boldsymbol{P}$ and $\boldsymbol{P} + span\{f\}$ are nonsingular,

$$dim(\boldsymbol{R}) - dim(\boldsymbol{P}) = dim(\boldsymbol{P} + span\{f\}) - dim(\boldsymbol{P}) \leq 1.$$

$\boldsymbol{P}$, $\boldsymbol{P} + span\{f\}$: nonsingular $\Rightarrow \boldsymbol{R}$: nonsingular $\Rightarrow \boldsymbol{P}$, $\boldsymbol{R}$: involutive

Case 1: $\boldsymbol{P} \subsetneq \boldsymbol{R}$.

$\Rightarrow$ $\exists$ a neighborhood $\boldsymbol{U}^0$ of x^0 and $z = \Phi(x)$ defined on $\boldsymbol{U}^0$ such that

$$span\{d\phi_{r+1}, \cdots, d\phi_n\} = \boldsymbol{R}^\perp$$

$$span\{d\phi_r, \cdots, d\phi_n\} = \boldsymbol{P}^\perp$$

on $\boldsymbol{U}^*$, where $r - 1 = dim(\boldsymbol{P})$.

$$f \in \boldsymbol{R} \quad \text{and} \quad g_i \in \boldsymbol{P} \subset \boldsymbol{R}$$

$$\Downarrow$$

$$\dot{z}_1 = f_1(z_1, \cdots, z_n) + g_1(z_1, \cdots, z_n)u$$

$$\vdots$$

$$\dot{z}_{r-1} = f_{r-1}(z_1, \cdots, z_n) + g_{r-1}(z_1, \cdots, z_n)u$$

$$\dot{z}_r = f_r(z_r, \cdots, z_n)$$

$$\dot{z}_{r+1} = 0$$

$$\vdots$$

$$\dot{z}_n = 0$$

Case 2: $\boldsymbol{P} = \boldsymbol{R}$.

$$f \in \boldsymbol{P} \subset \boldsymbol{R}$$

$$\Downarrow$$

$$\dot{z}_r = 0$$

Cases 1 and 2 $\Rightarrow$ All but (at most) one of the last $n - r + 1$ components that are not affected by input are constant with time.

Theorem: Suppose $\boldsymbol{R}$ with dimension r is nonsingular. Then, for each $x^0 \in \boldsymbol{U}$, $\exists$ a neighborhood $\boldsymbol{U}^0$ of x^0 and $z = \Phi(x)$ defined on $\boldsymbol{U}^0$ such that

a) the set $\mathcal{R}(x^0)$ of states reachable starting from x^0 along trajectories entirely contained in $\boldsymbol{U}^0$ is a subset of the slice

$$S_{x^0} = \{x \in \boldsymbol{U}^0 : \phi_{r+1}(x) = \phi_{r+1}(x^0), \cdots, \phi_n(x) = \phi_n(x^0)\}$$

b) $\mathcal{R}(x^0)$ contains an open subset of S_{x^0}

Proof of a) is obvious.

Theorem: Suppose P with dimension p and $P+span\{f\}$ are nonsingular. Then, for each $x^0 \in U$, $\exists$ a neighborhood U^0 of x^0 and $z = \Phi(x)$ defined on U^0 such that

a) the set $\mathcal{R}(x^0, T)$ of states reachable at time T starting from x^0 along trajectories entirely contained in U^0 is a subset of the slice

$$S_{x^0,T} = \{x \in U^0 : \phi_{p+1}(x) = \phi_{p+1}(\Phi^f_T(x^0)), \phi_{p+2}(x) = \phi_{p+2}(x^0), \cdots, \phi_n(x) = \phi_n(x^0)\}$$

where $\Phi^f_T(x^0)$ denotes the state reached at time $t = T$ when $u(t) = 0$ for all $t \in [0, T]$

b) $\mathcal{R}(x^0, T)$ contains an open subset of $S_{x^0,T}$

Proof of a) is obvious.

The system is locally controllable iff P is nonsingular with dimension n.

For single input case,

$$P = span\{g, ad_f g, \cdots, ad_f^{n-1} g\}.$$

Hence local controllability $\Leftrightarrow$ $dim[span\{g, ad_f g, \cdots, ad_f^{n-1} g\}] = n$

2.3 Local Observability

Proposition: Let Δ be a nonsingular involutive distribution of dimension d and suppose Δ is invariant under f, g_1, $\cdots$, g_m. Moreover, suppose $span\{dh_1, \cdots, dh_p\} \subset \Delta^\perp$. Then for each x^0, then $\exists$ a neighborhood U^0 of x^0 and $z = \Phi(x)$ defined on U^0 such that

$$\dot{\zeta}_1 = f_1(\zeta_1, \zeta_2) + g_1(\zeta_1, \zeta_2)u_i$$

$$\dot{\zeta}_2 = f_2(\zeta_2) + g_2(\zeta_2)u$$

$$y = h(\zeta_2)$$

where $\zeta_1 = (z_1, \cdots, z_d)$ and $\zeta_2 = (z_{d+1}, \cdots, z_n)$.

Proof: Using the coordinate transformation in Lemma (*), everything except $y = h(\zeta_2)$ is obvious. However, $span\{dh_1, \cdots, dh_p\} \subset \Delta^\perp$ implies an element of $span\{dh_1, \cdots, dh_p\}$ has the form

$$(0 \ \cdots \ 0 \ \omega_{d+1}(z) \ \cdots \ \omega_n(z))$$

For local decomposition in the above proposition, $\triangle$ must

i) be nonsingular

ii) be involutive

iii) be contained in the codistribution $(span\{dh_1, \cdots, dh_p\})^\perp$

iv) be invariant under $f, g_1, \cdots, g_m$

$$\Updownarrow$$

$\Omega = \triangle^\perp$ must

i') be nonsingular

ii') be spanned, locally around each point $x \in U$, by $n - d$ exact covector fields

iii') contains the codistribution $\bar{\Omega} = span\{dh_1, \cdots, dh_p\}$

iv') be invariant under $f, g_1, \cdots, g_m$

Question: Maximal such $\triangle$ or minimal such Ω?

Lemma: Let $\bar{\Omega}$ be a smooth codistribution and $\tau_1, \cdots, \tau_q$ vector fields. The family of all codistributions which are invariant under $\tau_1, \cdots, \tau_q$ and contain $\bar{\Omega}$ has a minimal element, which is a smooth codistribution.

$$\Downarrow$$

minimal Ω satisfying iii') and iv') always exists.

Question: Is minimal Ω satisfying iii') and iv') involutive.

Notation: $\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle$: The smallest codistribution that contains $\bar{\Omega}$ and is invariant under $\tau_1, \cdots, \tau_q$.

Lemma: Let

$$\Omega_0 = \bar{\Omega}$$

$$\Omega_k = \Omega_{k-1} + \sum_{i=1}^{q} L_{\tau_i} \Omega_{k-1}$$

Then, for all k,

$$\Omega_k \subset \langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle.$$

If $\exists k^*$ such that $\Omega_{k^*} = \Omega_{k^*+1}$, then

$$\Omega_{k^*} = \langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle.$$

Question: When can the stopping condition be met?

Lemma: $\exists$ an open and dense subset U^* of U such that, for each $x \in U^*$

$$\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle(x) = \Omega_{n-1}(x)$$

Lemma: Suppose $\bar{\Omega} = span\{d\lambda_1, \cdots, d\lambda_s\}$. Then $\exists$ an open and dense subset U^* of U with the following property. For each $x^0 \in U^*$, $\exists$ a neighborhood V of x^0 and d exact covector fields (with $d = dim\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle(x^0)$) $\omega_1, \cdots, \omega_d$ of the form

$$\omega_i = d\lambda_j \text{ or } \omega_i = dL_{v_r} \cdots L_{v_1} \lambda_j$$

where $r \leq n-1$ (which may depend on i) and $v_0, \cdots, v_r$ are vector fields in the set $\{\tau_1, \cdots, \tau_q\}$ and λ_j is a function in the set $\{\lambda_1, \cdots, \lambda_s\}$ such that

$$\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle(x) = span\{\omega_1(x), \cdots, \omega_d(x)\}$$

for all $x \in U^0$.

Lemma: Suppose $\bar{\Omega} = span\{d\lambda_1, \cdots, d\lambda_s\}$ and $\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle$ is nonsingular. Then $\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle^\perp$ is involutive.

Lemma: Suppose $\Omega = span\{d\lambda_1, \cdots, d\lambda_s\}$ and Ω_{n-1} is nonsingular. Then $\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle^\perp$ is involutive and

$$\langle \tau_1, \cdots, \tau_q | \bar{\Omega} \rangle = \Omega_{n-1}$$

To recap, let $\tau_1 = f$, $\tau_{i+1} = g_i$, $i = 1, \cdots, m$, and $\bar{\Omega} = span\{dh_1, \cdots, dh_p\}$. Then

$\langle f, g_1, \cdots, g_m | span\{dh_1, \cdots, dh_p\} \rangle^\perp$ nonsingular $\Rightarrow$ Local decompsition is possible.

Notation:

$$Q = \langle f, g_1, \cdots, g_m | span\{dh_1, \cdots, dh_p\} \rangle^\perp$$

Theorem: Suppose Q with dimension s is nonsingular. Then, for each $x^0 \in U$, $\exists$ a neighborhood U^0 of x^0 and $z = \Phi(x)$ defined on U^0 such that

a) Any two initial states x^a and x^b of U^0 such that

$$\phi_i(x^a) = \phi_i(x^b) \quad i = s+1, \cdots, n$$

produce identical output functions under any input which keeps the state trajectories evolving on U^0.

b) Any initial state x of U^0 which cannot be distinguished from x^0 belongs to the slice

$$S_{x^0} = \{x \in U^0 : \phi_i(x) = \phi_i(x^0), s+1 \leq i \leq n\}$$

The system is locally observable iff Q is nonsingular with dimension n.

Chapter 3

Exact Linearization for SISO Systems

Linear in Input System:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

3.1 Examples

Ex1:

$$\dot{x} = \frac{1}{1+x^2}u.$$

Let

$$y = x^3 + 3x$$
$$\Downarrow$$
$$x = \sqrt[3]{\frac{y}{2} + \sqrt{1 + \frac{y^2}{4}}} - \sqrt[3]{-\frac{y}{2} - \sqrt{1 + \frac{y^2}{4}}}.$$

Then

$$\dot{y} = (3x^2 + 3)\dot{x} = \frac{3 + 3x^2}{1 + x^2}u = 3u.$$

Ex2:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2^2 + (1 + x_1^2)u.$$

Let

$$u = \frac{1}{1 + x_1^2}(-x_2^2 + v).$$

$$\Downarrow$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = v.$$

Ex3:

$$\dot{x}_1 = x_1^2 + (1 + x_1^2)x_2$$

$$\dot{x}_2 = u.$$

Let

$$y_1 = x_1$$

$$y_2 = x_1^2 + (1 + x_1^2)x_2.$$

$$\Downarrow$$

$$x_1 = y_1$$

$$x_2 = \frac{y_2 - y_1^2}{1 + y_1^2}.$$

Hence

$$\dot{y}_1 = \dot{x}_1 = x_1^2 + (1 + x_1^2)x_2 = y_2$$

$$\dot{y}_2 = 2x_1\dot{x}_1 + 2x_1\dot{x}_1 x_2 + (1 + x_1^2)\dot{x}_2 = 2y_1y_2 + 2y_1y_2\frac{y_2 - y_1^2}{1 + y_1^2} + (1 + y_1^2)u$$

$$= 2y_1y_2\left(1 + \frac{y_2 - y_1^2}{1 + y_1^2}\right) + (1 + y_1^2)u.$$

Let

$$u = \frac{1}{1 + y_1^2}\left[-2y_1y_2\left(1 + \frac{y_2 - y_1^2}{1 + y_1^2}\right) + v\right].$$

$$\Downarrow$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = v.$$

3.2 Relative Degree

Def: A linear in input system is said to have relative degree r at a point x^0 if

i) $\boldsymbol{L}_g \boldsymbol{L}_f^k h(x) = 0$ for all x in a neighborhood of x^0 and all $k < r-1$

ii) $\boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x^0) \neq 0$

Warning: There may exist points where relative degree is not defined (Ex: $\boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x) \neq 0$ for all x in a neighborhood of x^0 but $\boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x^0) = 0$)

However, the relative degree is defined in an open and dense subset.

Linear System: $f(x) = \boldsymbol{A}x$, $g(x) = \boldsymbol{B}$, $h(x) = \boldsymbol{C}x$

$\Rightarrow$ $\boldsymbol{L}_f^k h(x) = \boldsymbol{C}\boldsymbol{A}^k x \Rightarrow$ $\boldsymbol{L}_g \boldsymbol{L}_f^k h(x) = \boldsymbol{C}\boldsymbol{A}^k \boldsymbol{B}$

$\Rightarrow$

$$\boldsymbol{C}\boldsymbol{A}^k \boldsymbol{B} = 0 \quad \forall k < r-1$$

$$\boldsymbol{C}\boldsymbol{A}^{r-1} \boldsymbol{B} \neq 0$$

Such r is the difference between degrees of denominator and numerator polynomials of the transfer function $\boldsymbol{H}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$ (see Problem 2.2-16 in Linear Systems by Kailath).

<u>Interpretation of Relative Degree</u>

Given $x(t^0) = x^0$,

$$y(t^0) = h(x(t^0)) = h(x^0)$$

$$y'(t) = \frac{dh}{dx}\frac{dx}{dt} = \frac{dh}{dx}[f(x(t)) + g(x(t))u(t)] = \boldsymbol{L}_f h(x(t)) + \boldsymbol{L}_g h(x(t))u(t)$$

If $r > 1$, $\boldsymbol{L}_g h(x(t)) = 0$ for all $t \approx t^0 \Rightarrow y'(t) = \boldsymbol{L}_f h(x(t)) \Rightarrow$

$$y''(t) = \frac{d\boldsymbol{L}_f h}{dx}\frac{dx}{dt} = \frac{d\boldsymbol{L}_f h}{dx}[f(x(t)) + g(x(t))u(t)] = \boldsymbol{L}_f^2 h(x(t)) + \boldsymbol{L}_g \boldsymbol{L}_f h(x(t))u(t)$$

If $r > 2$, $\boldsymbol{L}_g \boldsymbol{L}_f h(x(t)) = 0$ for all $t \approx t^0 \Rightarrow y''(t) = \boldsymbol{L}_f^2 h(x(t))$

By induction,

$$y^{(k)}(t) = \boldsymbol{L}_f^{k-1} h(x(t)) \;\; \forall k < r, t \approx t^0$$

$$y^{(r)}(t^0) = \boldsymbol{L}_f^r h(x^0) + \boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x^0) u(t^0)$$

$\Rightarrow$ r is exactly the number of times one has to differentiate the output $y(t)$ at time $t = t^0$ in order to have the value $u(t^0)$ of the input explicitly appearing.

A Technical Lemma: Let ϕ be a real valued function and f, g vector fields defined on $U \subset \mathbf{R}^n$. Then for any choice of $s, k, r \geq 0$

$$L_{ad_f^{k+r}g} L_f^s \phi(x) = \langle dL_f^s \phi(x), ad_f^{k+r} g(x) \rangle$$

$$= \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} \langle dL_f^{s+i} \phi(x), ad_f^k g(x) \rangle = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} L_{ad_f^k g} L_f^{s+i} \phi(x)$$

As a consequence,

i) $L_g \phi(x) = L_g L_f \phi(x) = \cdots = L_g L_f^k \phi(x) = 0 \;\; \forall x \in U$

ii) $L_g \phi(x) = L_{ad_f g} \phi(x) = \cdots = L_{ad_f^k g} \phi(x) = 0 \;\; \forall x \in U$

are equivalent.

Proof:

$$L_{ad_f^{k+r+1}g} L_f^s \phi(x) = L_{[f, ad_f^{k+r} g]} L_f^s \phi(x) = L_f L_{ad_f^{k+r}g} L_f^s \phi(x) - L_{ad_f^{k+r}g} L_f^{s+1} \phi(x)$$

$$= L_f \left[L_f L_{ad_f^{k+r-1}g} L_f^s \phi(x) - L_{ad_f^{k+r-1}g} L_f^{s+1} \phi(x) \right]$$

$$- \left[L_f L_{ad_f^{k+r-1}g} L_f^{s+1} \phi(x) - L_{ad_f^{k+r-1}g} L_f^{s+2} \phi(x) \right]$$

$$= L_f^2 L_{ad_f^{k+r-1}g} L_f^s \phi(x) - 2 L_f L_{ad_f^{k+r-1}g} L_f^{s+1} \phi(x) + L_{ad_f^{k+r-1}g} L_f^{s+2} \phi(x)$$

$$\vdots$$

Now the first part follows.

For the proof of the second part, let $s = k = 0$. Then,

$$L_{ad_f^r g} \phi(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} L_g L_f^i \phi(x)$$

Hence, i) $\Rightarrow$ ii). On the other hand,

$$L_g L_f^r \phi = L_f L_g L_f^{r-1} \phi - L_{[f,g]} L_f^{r-1} \phi$$

$$= L_f^2 L_g L_f^{r-2} \phi - 2 L_f L_{[f,g]} L_f^{r-2} \phi + L_{[f,[f,g]]} L_f^{r-2} \phi$$

$$= L_f^2 L_g L_f^{r-2} \phi - 2 L_f L_{ad_f g} L_f^{r-2} \phi + L_{ad_f^2 g} L_f^{r-2} \phi$$

$$\vdots$$

Hence,

$$L_g L_f^r g(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_f^{r-i} L_{ad_f^i g} \phi(x)$$

Hence, ii) $\Rightarrow$ i).

Lemma: $dh(x^0)$, $dL_f h(x^0)$, $\cdots$, $dL_f^{r-1} h(x^0)$ are linearly independent.

Proof: $\phi = h$, $s = j$, $k = 0$, $r = i$ in the previous lemma $\Rightarrow$

$$\langle dL_f^j h(x), ad_f^i g(x) \rangle = \sum_{l=0}^{i} (-1)^l \binom{i}{l} L_f^{i-l} L_g L_f^{j+l} h(x) = 0 \quad \text{for } x \approx x^0,\ i+j \leq r-2$$

$$\langle dL_f^j h(x^0), ad_f^i g(x^0) \rangle = (-1)^{r-1-j} \begin{pmatrix} i \\ j \end{pmatrix} L_g L_f^{r-1} h(x^0) \neq 0 \quad \forall i + j = r - 1$$

$$\Downarrow$$

$$\begin{bmatrix} dh(x^0) \\ dL_f h(x^0) \\ \vdots \\ dL_f^{r-1} h(x^0) \end{bmatrix} [g(x^0)\ ad_f g(x^0)\ \cdots\ ad_f^{r-1} g(x^0)]$$

$$= \begin{bmatrix} 0 & \cdots & \langle dh(x^0), ad_f^{r-1} g(x^0) \rangle \\ 0 & \cdots & * \\ \vdots & \cdots & * \\ \langle dL_f^{r-1} h(x^0), g(x^0) \rangle & * & * \end{bmatrix} \qquad (***)$$

has rank r and, thus, the lemma follows.

$\Rightarrow$ $r \leq n$ since $n+1$ covectors cannot be linearly independent in n-D space.

Proposition: Suppose the system has relative degree r at x^0. Then $r \leq n$.

Set

$$\phi_1(x) = h(x)$$
$$\phi_2(x) = L_f h(x)$$
$$\vdots$$
$$\phi_r(x) = L_f^{r-1} h(x).$$

If $r < n$, $\exists$ $n-r$ functions $\phi_{r+1}(x), \cdots, \phi_n(x)$ such that

$$\Phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{bmatrix}$$

has nonsingular Jacobian at x^0 and thus defines a coordinate transformation in a neighborhood of x^0. The value at x^0 of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\phi_{r+1}(x), \cdots, \phi_n(x)$ in such a way that

$$L_g \phi_i(x) = 0 \;\; \forall r+1 \leq i \leq n, x \approx x^0.$$

Proof: $L_g L_f^{r-1} h(x^0) \neq 0 \Rightarrow g(x^0) \neq 0 \Rightarrow G = span\{g\}$ is nonsingular around x^0.

G: 1-D $\Rightarrow$ G: involutive $\Rightarrow$ $\exists$ $\lambda_1(x), \cdots, \lambda_{n-1}(x)$ such that

$$span\{d\lambda_1, \cdots, d\lambda_{n-1}\} = G^{\perp}.$$

Suppose

$$dim(G^{\perp} + span\{dh, dL_f h, \cdots, dL_f^{r-1} h\}) \neq n$$

$$\Downarrow$$

$$G^{\perp} \supset span\{dh, dL_f h, \cdots, dL_f^{r-1} h\}$$

$$\Downarrow$$

$$(span\{dh, dL_f h, \cdots, dL_f^{r-1} h\})^{\perp} \supset G$$

$$\Downarrow$$

$g(x^0)$ annihilates all covectors in $span\{dh, dL_f h, \cdots, dL_f^{r-1} h\}(x^0)$

$$\Downarrow$$

Contradiction (because $\langle dL_f^{r-1} h(x^0), g(x^0) \rangle \neq 0$)

$$\Downarrow$$

$$dim(G^{\perp} + span\{dh, dL_f h, \cdots, dL_f^{r-1} h\}) = n$$

$$\Downarrow$$

$\exists\ n-r$ functions among $\{\lambda_1, \cdots, \lambda_{n-1}\}$, WLOG $\lambda_1, \cdots, \lambda_{n-r}$, such that

$$dh, d\boldsymbol{L}_f h, \cdots, d\boldsymbol{L}_f^{r-1} h, d\lambda_1, \cdots, d\lambda_{n-r}$$

are linearly independent around x^0. By construction,

$$\langle d\lambda_i(x), g(x)\rangle = \boldsymbol{L}_g \lambda_i(x) = 0 \ \ \forall 1 \le i \le n-r, x \approx x^0.$$

Note that $\lambda_i(x)$ can be replaced with $\lambda_i(x) + c_i$ and, thus, WLOG $\lambda_i(x^0)$ can be assigned arbitrarily.

$$z = \Phi(x)$$
$$\Downarrow$$
$$\frac{dz_1}{dt} = \frac{d\phi_1}{dx}\frac{dx}{dt} = \frac{dh}{dx}\frac{dx}{dt} = \boldsymbol{L}_f h(x(t)) = \phi_2(x(t)) = z_2(t)$$
$$\vdots$$
$$\frac{dz_{r-1}}{dt} = \frac{d\phi_{r-1}}{dx}\frac{dx}{dt} = \frac{d(\boldsymbol{L}_f^{r-2} h)}{dx}\frac{dx}{dt} = \boldsymbol{L}_f^{r-1} h(x(t)) = \phi_r(x(t)) = z_r(t)$$
$$\frac{dz_r}{dt} = \boldsymbol{L}_f^r h(x(t)) + \boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x(t)) u(t) = b(z(t)) + a(z(t))u(t)$$

where

$$a(z) = \boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(\Phi^{-1}(z))$$
$$b(z) = \boldsymbol{L}_f^r h(\Phi^{-1}(z)).$$

Note: $a(z^0 = \Phi(x^0)) \neq 0 \Rightarrow a(z) \neq 0$ for all $z \approx z^0$.

Suppose $\phi_{r+1}(x), \cdots, \phi_n(x)$ are chosen in such a way that $\boldsymbol{L}_g \phi_i(x) = 0$. Then

$$\frac{dz_i}{dt} = \frac{d\phi_i}{dx}(f(x(t)) + g(x(t))u(t)) = \boldsymbol{L}_f \phi_i(x(t)) + \boldsymbol{L}_g \phi_i(x(t))u(t)$$
$$= \boldsymbol{L}_f \phi_i(x(t)) = q_i(z(t))$$

where

$$q_i(z) = \boldsymbol{L}_f \phi_i(\Phi^{-1}(z)) \ \ \forall r+1 \le i \le n.$$

To recap,

$$\frac{dz_1}{dt} = z_2(t)$$

$$\frac{dz_2}{dt} = z_3(t)$$

$$\vdots$$

$$\frac{dz_{r-1}}{dt} = z_r(t)$$

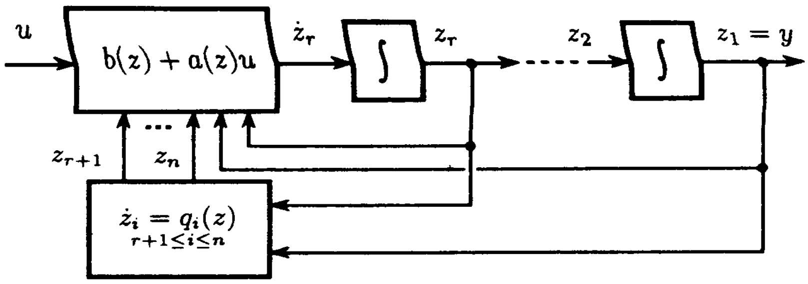
$$\frac{dz_r}{dt} = b(z(t)) + a(z(t))u(t)$$

$$\frac{dz_{r+1}}{dt} = q_{r+1}(z(t))$$

$$\vdots$$

$$\frac{dz_n}{dt} = q_n(z(t))$$

$$y = h(x) = z_1.$$



3.3 Exact Linearization by Feedback

Two systems

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u, \quad x_1 \in \mathbf{R}^n$$

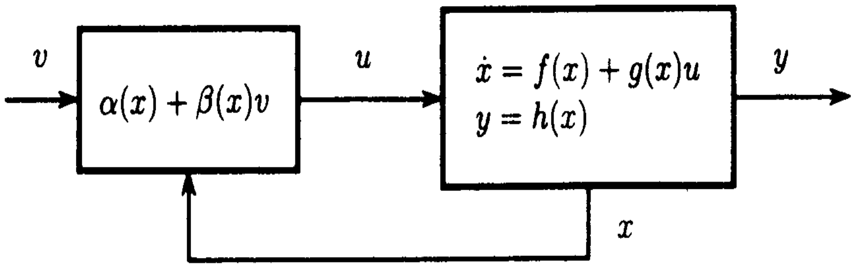
$$\dot{x}_2 = f_2(x_2) + g_2(x_2)u, \quad x_2 \in \mathbf{R}^n$$

are said to be locally feedback equivalent if $\exists\ u = \alpha(x)+\beta(x)v$ and $x_2 = \Phi(x_1)$ such that the closed loop system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha(x_1) + g_1(x_1)\beta(x_1)v$$

in x_2 coordinate is

$$\begin{aligned}\dot{x}_2 &= \left[\frac{d\Phi}{dx_1}(f_1 + g_1\alpha)\right]_{x_1=\Phi^{-1}(x_2)} + \left[\frac{d\Phi}{dx_1}(g_1\beta)\right]_{x_1=\Phi^{-1}(x_2)} v \\ &= f_2(x_2) + g_2(x_2)v.\end{aligned}$$



Question: Is the given nonlinear system feedback equivalent to a linear and controllable form.

A linear in input system is said to be locally state feedback linearisable if it is locally feedback equivalent to a linear system in (linear and controllable) Brunovsky controller form

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v = A_c z + b_c v.$$

Suppose $r = n$.

$$z = \Phi(x) = \begin{bmatrix} h(x) \\ \boldsymbol{L}_f h(x) \\ \vdots \\ \boldsymbol{L}_f^{n-1} h(x) \end{bmatrix}$$

$$\Downarrow$$

$$\frac{dz_1}{dt} = z_2(t)$$

$$\frac{dz_2}{dt} = z_3(t)$$

$$\vdots$$

$$\frac{dz_{n-1}}{dt} = z_n(t)$$

$$\frac{dz_n}{dt} = b(z(t)) + a(z(t))u(t).$$

Note: $a(z) \neq 0$ for all $z \approx z^0 = \Phi^{-1}(x^0)$

$$u = \frac{1}{a(z)}(-b(z) + v) = \frac{1}{\boldsymbol{L}_g \boldsymbol{L}_f^{n-1} h(x)}(-\boldsymbol{L}_f^n h(x) + v)$$

$$\Downarrow$$

$$\frac{dz_1}{dt} = z_2(t)$$

$$\frac{dz_2}{dt} = z_3(t)$$

$$\vdots$$

$$\frac{dz_{n-1}}{dt} = z_n(t)$$

$$\frac{dz_n}{dt} = v(t).$$

$\Rightarrow$ Brunovsky controller form $\Rightarrow$ State Feedback Linearisable.

$f(x^0) = 0$ (x^0: equilibrium point) and $h(x^0) = 0$ (WLOG) $\Rightarrow$ $z^0 = \Phi(x^0) = 0$ (equilibrium point in new coordinate)

Lemma: The exact linearization is possible iff $\exists$ a neighborhood U of x^0 and a real valued function $\lambda(x)$, defined on U, such that

$$\dot{x} = f(x) + g(x)u$$

$$y = \lambda(x)$$

has relative degree n at x^0.

The proof of sufficiency is obvious.

Question: How to find λ?

Answer:

Solve PDEs

$$L_g\lambda(x) = L_g L_f \lambda(x) = \cdots = L_g L_f^{n-2}\lambda(x) = 0$$

with

$$L_g L_f^{n-1}\lambda(x^0) \neq 0.$$

This condition prevents trivial solution $\lambda(x) = c$.

$$\Updownarrow$$

Solve ODEs

$$L_g\lambda(x) = L_{ad_f g}\lambda(x) = \cdots = L_{ad_f^{n-2} g}\lambda(x) = 0$$

with

$$L_{ad_f^{n-1} g}\lambda(x^0) \neq 0.$$

Theorem: TFAE

a) The exact feedback linearisation is possible near x^0.

b) $\exists$ a real valued function $\lambda(x)$ defined in a neighborhood U of x^0 solving the PDEs with the condition.

c)

i) $(g(x^0)\ ad_f g(x^0)\ \cdots\ ad_f^{n-1} g(x^0))$ has rank n

ii) $D = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive in a neighborhood of x^0.

Proof: (b $\Rightarrow$ c) (***) $\Rightarrow$ i)

$\Rightarrow$ D nonsingular

$$L_g\lambda(x) = L_{ad_f g}\lambda(x) = \cdots = L_{ad_f^{n-2} g}\lambda(x) = 0$$

$$\Updownarrow$$

$$d\lambda(x)\left(g(x)\ ad_f g(x)\ \cdots\ ad_f^{n-2} g(x)\right) = 0$$

$\Rightarrow$ $\boldsymbol{D}$ is completely integrable $\Rightarrow$ ii).

(b $\Leftarrow$ c) i) $\Rightarrow$ $\boldsymbol{D}$ nonsingular with dimension $n-1$

ii) $\Rightarrow$ $\exists$ $\lambda(x)$ such that $\boldsymbol{D}^{\perp} = span\{d\lambda\}$

$\Rightarrow$ $\lambda(x)$: solution of PDE.

Suppose $\lambda(x)$ doesn't satisfy the condition.

$\Rightarrow$ $d\lambda(x)$ is annihilated by a set of n linearly independent vectors $\Rightarrow$ $d\lambda(x) = 0$ $\Rightarrow$ contradiction.

$\Rightarrow$ $\lambda(x)$ does satisfy the condition.

Note: For 2D systems, ii) is always satisfied.

The stabilisation of state feedback linearisable system can be achieved by additional pole placing static state feedback

$$v = -\boldsymbol{K}z = -c_0 h(x) - c_1 \boldsymbol{L}_f h(x) - \cdots - c_{n-1}\boldsymbol{L}_f^{n-1} h(x)$$

where

$$\boldsymbol{K} = (c_0 \cdots c_{n-1})$$

$$\Downarrow$$

$$u = \frac{-\boldsymbol{L}_f^n h(x) - \sum_{i=0}^{n-1} c_i \boldsymbol{L}_f^i h(x)}{\boldsymbol{L}_g \boldsymbol{L}_f^{n-1} h(x)}.$$

3.4 Exact Partial or Input/Output Linearization by Feedback

Suppose $r < n$ for any λ.

Full linearisation is impossible but partial linearisation whose input/output representation is linear is possible.

$$u = \frac{1}{a(z)}(-b(z) + v) = \frac{1}{\boldsymbol{L}_g \boldsymbol{L}_f^{r-1} h(x)}(-\boldsymbol{L}_f^r h(x) + v)$$

$$\Downarrow$$

$$\frac{dz_1}{dt} = z_2(t)$$

$$\frac{dz_2}{dt} = z_3(t)$$

$$\vdots$$

$$\frac{dz_{r-1}}{dt} = z_r(t)$$

$$\frac{dz_r}{dt} = v(t)$$

$$\frac{dz_{r+1}}{dt} = q_{r+1}(z(t))$$

$$\vdots$$

$$\frac{dz_n}{dt} = q_n(z(t))$$

$$y = z_1$$

or

$$\dot{\xi} = A_c\xi + b_c v$$

$$\dot{\eta} = q(\xi, \eta)$$

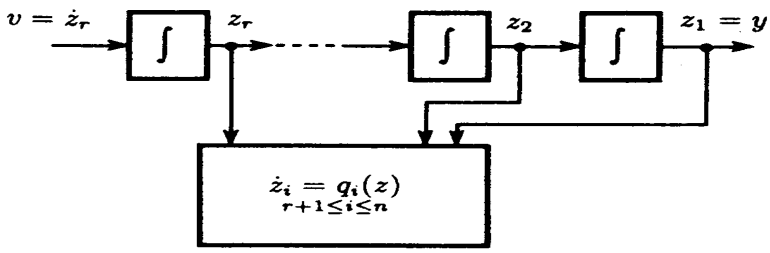
$$y = z_1$$

where

$$\xi = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}, \quad \eta = \begin{bmatrix} z_{r+1} \\ \vdots \\ z_n \end{bmatrix}, \quad q = \begin{bmatrix} q_{r+1} \\ \vdots \\ q_n \end{bmatrix}.$$

$$\Downarrow$$

The input/output behaviour of the system is completely characterised by the r-D linear subsystem: $H(s) = \frac{1}{s^r}$



3.5 Zero Dynamics

Consider the exact input-output linearising control law for a linear system: $f(x) = Ax, g(x) = b, h(x) = c^T x$. Suppose A, b, c is a minimal realisation. Now the transfer function from v to y is $\frac{1}{s^r}$ and, thus, the exact input-output linearising control cancels the zeros of the original system and places the rest of the closed loop system poles at the origin. Hence the exact input-output linearising control law is the nonlinear counterpart of the zero cancelling control law.

Notice that if there was unstable zeros, the resulting closed loop system is unstable due to the unstable poles that cancels the unstable zeros.

Output Zeroing Problem: Find initial state x_0 and input u such that $y = 0$ for all $t \geq 0$.

$y \equiv 0 \Leftrightarrow \xi_1 = \cdots = \xi_r \equiv 0$.

Clearly this is achieved if $\xi_1(0) = \cdots = \xi_r(0) = 0$ and

$$u_0(t) = -\frac{b(0, \eta(t))}{a(0, \eta(t))}$$

where η is any solution of the zero dynamics:

$$\dot{\eta} = q(0, \eta), \qquad \eta(0) \text{ arbitrary.}$$

The name of zero dynamics stems from the fact that it is the internal dynamics with the constraints that $y \equiv 0$. Indeed for linear systems, the denominator polynomial of the zero dynamics is the numerator polynomial of the original system (see Remark 4.3.1 in the text book).

For output zeroing in x coordinate, we need

$$x_0 \in M := \{x : h(x) = \cdots = L_f^{r-1} h(x) = 0\}$$

and

$$u_0(t) = -\frac{b(0, \eta(x(t)))}{a(0, \eta(x(t)))}.$$

Then the zero dynamics is the restriction of

$$\dot{x} = f(x) + g(x) u_0$$

to M.

Note if $z = 0$ is the equilibrium point, $\eta = 0$ is the equilibrium of the zero dynamics.

Def.: The linear in input system is locally asymptotically (exponentially) minimum phase at x_0 if the equilibrium point $\eta = 0$ of the zero dynamics is locally asymptotically (exponentially) stable.

Remark: If the system under consideration have multiple equilibrium, it may be minimum phase at some equilibrium points and be nonminimum phase at some others. Moreover some equilibrium points of the system may not be equilibrium points of the zero dynamics.

Theorem (Stabilization of Minimum Phase Systems): If the system has relative degree r and is locally exponentially minimum phase, then

$$v = -\boldsymbol{K}z = -c_0 h(x) - c_1 \boldsymbol{L}_f h(x) - \cdots - c_{r-1} \boldsymbol{L}_f^{r-1} h(x)$$

with $s^r + c_{r-1}s^{r-1} + \cdots + c_0$ Hurwitz results in a locally exponentially stable closed loop system.

Proof: The closed loop system has the form

$$\dot{\xi} = \boldsymbol{A}\xi$$

$$\dot{\eta} = q(\xi, \eta)$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{r-1} \end{bmatrix}.$$

The linearization at the origin is

$$\begin{bmatrix} \boldsymbol{A} & 0 \\ \frac{\partial q}{\partial \xi}(0,0) & \frac{\partial q}{\partial \eta}(0,0) \end{bmatrix}.$$

By assumption, $\boldsymbol{A}$ is Hurwitz. Moreover, exponential minimum phase implies $\frac{\partial q}{\partial \eta}(0,0)$ is also Hurwitz. Hence the theorem follows.

3.6 Exact Linearization by Change of Coordinates

Two systems

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u, \quad x_1 \in \mathbf{R}^n$$
$$\dot{x}_2 = f_2(x_2) + g_2(x_2)u, \quad x_2 \in \mathbf{R}^n$$

are said to be locally state equivalent if $\exists\ x_2 = \Phi(x_1)$ such that

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u$$

in x_2 coordinate is

$$\begin{aligned}\dot{x}_2 &= \left[\frac{d\Phi}{dx_1} f_1\right]_{x_1=\Phi^{-1}(x_2)} + \left[\frac{d\Phi}{dx_1} g_1\right]_{x_1=\Phi^{-1}(x_2)} u \\ &= f_2(x_2) + g_2(x_2)u\end{aligned}$$

Question: Is the given nonlinear system state equivalent to a linear and controllable form.

A linear in input system is said to be locally state linearisable if it is locally state equivalent to a linear controllable system of the form

$$\dot{z} = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \cdots & 0 \\ -\alpha_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\alpha_1 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & 0 & 0 & \cdots & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v$$

Lemma: Let $z = \Phi(x)$. Then

$$\frac{d(\Phi^{-1})}{dz} = \left(\left[\frac{d\Phi}{dx}\right]_{x=\Phi^{-1}(z)}\right)^{-1}$$

Proof:

$$z = \Phi \circ \Phi^{-1}(z)$$
$$\Downarrow$$

$$I = \left[\frac{d\Phi}{dx}\right]_{x=\Phi^{-1}(z)} \frac{d(\Phi^{-1})}{dz}$$

Lemma: Let $z = \Phi(x)$. Then $[f, g]$ in z coordinate is equal to $[\bar{f}, \bar{g}]$ where

$$\bar{f} = \left[\frac{d\Phi}{dx} f\right]_{x=\Phi^{-1}(z)} \quad \bar{g} = \left[\frac{d\Phi}{dx} g\right]_{x=\Phi^{-1}(z)}$$

Proof:

$$\begin{aligned}
[\bar{f}, \bar{g}] &= \left[\left[\frac{d\Phi}{dx} f\right]_{x=\Phi^{-1}(z)}, \left[\frac{d\Phi}{dx} g\right]_{x=\Phi^{-1}(z)}\right] \\
&= \left(\left[\frac{d}{dx}\left[\frac{d\Phi}{dx} g\right]\right]_{x=\Phi^{-1}(z)} \frac{d(\Phi^{-1})}{dz}\right) \left[\frac{d\Phi}{dx} f\right]_{x=\Phi^{-1}(z)} \\
&\quad - \left(\left[\frac{d}{dx}\left[\frac{d\Phi}{dx} f\right]\right]_{x=\Phi^{-1}(z)} \frac{d(\Phi^{-1})}{dz}\right) \left[\frac{d\Phi}{dx} g\right]_{x=\Phi^{-1}(z)} \\
&= \left[\left(\frac{d}{dx}\left[\frac{d\Phi}{dx} g\right]\right) f\right]_{x=\Phi^{-1}(z)} - \left[\left(\frac{d}{dx}\left[\frac{d\Phi}{dx} f\right]\right) g\right]_{x=\Phi^{-1}(z)} \\
&= \left[\left(\frac{d^2\Phi}{dx^2} g + \frac{d\Phi}{dx}\frac{dg}{dx}\right) f\right]_{x=\Phi^{-1}(z)} - \left[\left(\frac{d^2\Phi}{dx^2} f + \frac{d\Phi}{dx}\frac{df}{dx}\right) g\right]_{x=\Phi^{-1}(z)} \\
&= \left[\frac{d\Phi}{dx}\left(\frac{dg}{dx} f - \frac{df}{dx} g\right)\right]_{x=\Phi^{-1}(z)} = \left[\frac{d\Phi}{dx}[f, g]\right]_{x=\Phi^{-1}(z)}
\end{aligned}$$

Theorem: TFAE

a) The exact state linearization is possible near x^0.

b)

i) $(g(x^0)\ ad_f g(x^0)\ \cdots\ ad_f^{n-1} g(x^0))$ has rank n

ii)$[ad_f^i g, ad_f^j g] = 0$, $0 \le i, j \le n$, or, equivalently $[g, ad_f^i g] = 0$, $0 \le i \le 2n-1$.

Proof: (a $\Rightarrow$ b) Obvious since i) and ii) hold for the given form of linear controllable system and are invariant under coordinate transformations.

(a $\Leftarrow$ b) WLOG, we assume $f(0) = 0$ (0 is the equil. pt of the original system). Let

$$D_0 = span\{ad_f g, \cdots, ad_f^{n-1} g\}$$

$$D_i = span\{g, \cdots, ad_f^{i-1}g, ad_f^{i+1}g, \cdots, ad_f^{n-1}g\} \quad 1 \leq i \leq n-2$$

$$D_{n-1} = span\{g, \cdots, ad_f^{n-2}g\}$$

ii) and i) $\Rightarrow$ D_i is involutive and of rank $n-1$
$\Rightarrow$ $\exists$ $\bar{\phi}_i$, $1 \leq i \leq n$, such that

$$\langle d\bar{\phi}_i, D_{n-i}\rangle = 0$$

$$\gamma_i := \langle d\bar{\phi}_i, ad_{(-f)}^{n-i}g\rangle \neq 0$$

i) $\Rightarrow$ $rank\{d\bar{\phi}_1, \cdots, d\bar{\phi}_n\} = n$
$\Rightarrow$ $\bar{z} = (\bar{\phi}_1(x) \ \cdots \ \bar{\phi}_n(x))^T =: \bar{\Phi}(x)$ local diffeomorphism
In $\bar{z}$ coordinate,

$$\bar{g}(\bar{z}) := \left[\frac{d\bar{\Phi}}{dx}g\right]_{x=\bar{\Phi}^{-1}(\bar{z})} = \gamma_n \circ \bar{\Phi}^{-1}(\bar{z})e_n =: \bar{\gamma}_n(\bar{z})e_n$$

$$ad_{(-\bar{f})}\bar{g}(\bar{z}) = \left[\frac{d\bar{\Phi}}{dx}ad_{(-f)}g\right]_{x=\bar{\Phi}^{-1}(\bar{z})} = \gamma_{n-1} \circ \bar{\Phi}^{-1}(\bar{z})e_{n-1} =: \bar{\gamma}_{n-1}(\bar{z})e_{n-1}$$

$$\vdots$$

$$ad_{(-\bar{f})}^{n-1}\bar{g}(\bar{z}) = \left[\frac{d\bar{\Phi}}{dx}ad_{(-f)}^{n-1}g\right]_{x=\bar{\Phi}^{-1}(\bar{z})} = \gamma_1 \circ \bar{\Phi}^{-1}(\bar{z})e_1 =: \bar{\gamma}_1(\bar{z})e_1$$

where

$$\bar{f}(\bar{z}) := \left[\frac{d\bar{\Phi}}{dx}f\right]_{x=\bar{\Phi}^{-1}(\bar{z})}$$

ii) $\Rightarrow$ $[\bar{\gamma}_i(\bar{z})e_i, \bar{\gamma}_j(\bar{z})e_j] = \langle d\bar{\gamma}_j, \bar{\gamma}_i e_i\rangle e_j - \langle d\bar{\gamma}_i, \bar{\gamma}_j e_j\rangle e_i = 0$
$\Rightarrow$ $\langle d\bar{\gamma}_j, \bar{\gamma}_i e_i\rangle = \langle d\bar{\gamma}_i, \bar{\gamma}_j e_j\rangle = 0 \Rightarrow \frac{d\bar{\gamma}_j}{d\bar{z}_i}\bar{\gamma}_i = \frac{d\bar{\gamma}_i}{d\bar{z}_j}\bar{\gamma}_j = 0$
$\Rightarrow$ $\frac{d\bar{\gamma}_j}{d\bar{z}_i} = \frac{d\bar{\gamma}_i}{d\bar{z}_j} = 0 \Rightarrow \bar{\gamma}_i = \bar{\gamma}_i(\bar{z}_i)$
Define $\delta_i(\bar{z}_i) = \int_0^{\bar{z}_i} \frac{1}{\bar{\gamma}_i(\xi)}d\xi$ and $\phi_i = \delta_i \circ \bar{\phi}_i$ so that for $1 \leq i \leq n$,

$$\langle d\phi_i, D_{n-i}\rangle = 0$$

$$\langle d\phi_i, ad_{(-f)}^{n-i}g\rangle = 1$$

and $z = (\phi_1(x) \ \cdots \ \phi_n)^T =: \Phi(x)$ local diffeomorphism

In z coordinate,

$$\tilde{g}(z) := \left[\frac{d\Phi}{dx} g\right]_{x=\Phi^{-1}(z)} = e_n$$

$$ad_{(-f)}\tilde{g} := \left[\frac{d\Phi}{dx} ad_{(-f)} g\right]_{x=\Phi^{-1}(z)} = e_{n-1}$$

$$\vdots$$

$$ad^{n-1}_{(-f)}\tilde{g} := \left[\frac{d\Phi}{dx} ad^{n-1}_{(-f)} g\right]_{x=\Phi^{-1}(z)} = e_1$$

where

$$\tilde{f}(z) := \left[\frac{d\Phi}{dx} f\right]_{x=\Phi^{-1}(z)}$$

$$\Downarrow$$

$$e_{n-1} = ad_{-f}\tilde{g} = [-\tilde{f}, e_n] = \sum_{i=1}^{n} \frac{d\tilde{f}_i}{dz_n} e_i$$

$\Rightarrow \frac{d\tilde{f}_{n-1}}{dz_n} = 1$ and $\frac{d\tilde{f}_i}{dz_n} = 0,\ i \neq n-1$

$$e_{n-2} = ad^2_{-f}\tilde{g} = [-\tilde{f}, e_{n-1}] = \sum_{i=1}^{n} \frac{d\tilde{f}_i}{dz_{n-1}} e_i$$

$\Rightarrow \frac{d\tilde{f}_{n-2}}{dz_{n-1}} = 1$ and $\frac{d\tilde{f}_i}{dz_{n-1}} = 0,\ i \neq n-2$

$$\vdots$$

$$e_1 = ad^{n-1}_{-f}\tilde{g} = [-\tilde{f}, e_2] = \sum_{i=1}^{n} \frac{d\tilde{f}_i}{dz_2} e_i$$

$\Rightarrow \frac{d\tilde{f}_1}{dz_2} = 1$ and $\frac{d\tilde{f}_i}{dz_2} = 0,\ i \neq 1$

$$\Downarrow$$

$$\tilde{f}_i = \psi_i(z_1) + z_{i+1} \qquad 1 \leq i \leq n-1$$

$$\tilde{f}_n = \psi_n(z_1)$$

$$\Downarrow$$

$$ad^n_{(-f)}\tilde{g} = ad_{(-f)}ad^{n-1}_{(-f)}\tilde{g} = [-\tilde{f}, e_1] = \sum_{i=1}^{n}\frac{d\psi_i}{dz_1}(z_1)e_i$$

ii) implies

$$0 = [ad^{n-1}_{(-f)}\tilde{g}, ad^n_f\tilde{g}] = [e_1, ad^n_f\tilde{g}] = (-1)^n\sum_{i=1}^{n}\frac{d^2\psi_i}{dz_1^2}(z_1)e_i$$

$f(0) = 0 \Rightarrow \psi_i(z_1) = -\alpha_{n-i}z_1$

3.7 Observers with Linear Error Dynamics

For the sake of brevity, consider

$$\dot{x} = f(x)$$

$$y = h(x)$$

Goal: find a neighborhood U^0 of x^0 and $z = \Phi(x)$ defined on U^0, such that

$$\dot{z} = \left[\frac{d\Phi}{dx}f(x)\right]_{x=\Phi^{-1}(z)} = Az + k(Cz)$$

$$y = h(\Phi^{-1}(z)) = Cz$$

for all $z \in \Phi(U^0)$, $k : h(U^0) \to \mathbf{R}^n$, and (C, A) observable.

Suppose the goal is possible. Then consider an observer of the form

$$\dot{\xi} = (A + GC)\xi - Gy + k(y)$$

$$\Downarrow$$

Linear Error Dynamics

$$\dot{e} = (A + GC)e$$

where

$$e = \xi - z = \xi - \Phi(x)$$

Note that, from the observability of (C, A), the eigenvalues of $A + GC$ can be arbitrarily assigned.

Theorem: The goal is possible iff

i) $dim(span\{dh(x^0), d\boldsymbol{L}_f h(x^0), \cdots, d\boldsymbol{L}_f^{n-1} h(x^0)\}) = n$

ii) τ is such that

$$[ad_f^i \tau, ad_f^j \tau] = 0$$

for all $1 \leq i, j \leq n$.

Note: By Jacobi Indentity, ii) $\Leftrightarrow$

$$[\tau, ad_f^k \tau] = 0$$

for all $1 \leq k \leq 2n - 1$.

Chapter 4

Canonocal Forms

4.1 Parametric Strict-Feedback Canonical Form

Consider

$$\dot{\xi} = f_0(\xi) + \sum_{i=1}^{p} \theta_i f_i(\xi) + g_0(\xi) u$$

where $\theta_1, \cdots, \theta_p$ are constant unknown parameters.

Question: When do there exist a parameter independent control $u = \alpha(\xi) + \beta(\xi) v$ and $x = \Phi(\xi)$ such that the closed loop system in x coordinate is of the parametric strict feedback canonical form:

$$\dot{x}_1 = x_2 + \phi_1^T(x_1)\theta$$

$$\dot{x}_2 = x_3 + \phi_2^T(x_1, x_2)\theta$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}^T(x_1, \cdots, x_{n-1})\theta$$

$$\dot{x}_n = v + \phi_n^T(x)\theta$$

The parameter independence implies the feedback transformation must hold for $\theta = 0$.

For $\theta = 0$, the question boils down to the feedback linearisability of $\dot{\xi} = f_0(\xi) + g_0(\xi) u$

$\Rightarrow$ The feedback linearisability of $\dot{\xi} = f_0(\xi) + g_0(\xi)u$ is necessary for the question.

For the question, f_i in x coordinate must have the form

$$\bar{f}_i = \left[\frac{d\Phi}{dx} f_i\right]_{\xi = \Phi^{-1}(x)} = \begin{bmatrix} \phi_{1i}(x_1) \\ \phi_{2i}(x_1, x_2) \\ \vdots \\ \phi_{ni}(x) \end{bmatrix}$$

$$\Updownarrow$$

$$[e_n, \bar{f}_i] = \frac{\partial \phi_{ni}}{\partial x_n} e_n \in span\{e_n\}$$

$$[e_{n-1}, \bar{f}_i] = \frac{\partial \phi_{(n-1)i}}{\partial x_{n-1}} e_{n-1} + \frac{\partial \phi_{ni}}{\partial x_{n-1}} e_n \in span\{e_n, e_{n-1}\}$$

$$\vdots$$

$$[e_1, \bar{f}_i] = \sum_{j=1}^{n} \frac{\partial \phi_{ji}}{\partial x_1} e_j \in span\{e_n, e_{n-1}, \cdots, e_1\}$$

$$\Updownarrow$$

$$[\bar{g}_0, \bar{f}_i] = [ae_n, \bar{f}_i] = a[e_n, \bar{f}_i] - \boldsymbol{L}_{\bar{f}_i} a e_n \in span\{e_n\} = span\{\bar{g}_0\} = \bar{G}^0$$

$$[ad_{\bar{f}_0} \bar{g}_0, \bar{f}_i] = [[\boldsymbol{A}_c x + b e_n, a e_n], \bar{f}_i] = [[\boldsymbol{A}_c x, a e_n] + [b e_n, a e_n], \bar{f}_i]$$

$$= \left[\left(\sum_{i=1}^{n-1} \frac{\partial a}{\partial x_i} x_{i+1} \right) e_n - a e_{n-1} + b \frac{\partial a}{\partial x_n} e_n - a \frac{\partial b}{\partial x_n} e_n, \bar{f}_i \right] =: [\alpha e_n + \beta e_{n-1}, \bar{f}_i]$$

$$=: [\alpha e_n, \bar{f}_i] + [\beta e_{n-1}, \bar{f}_i] = \alpha[e_n, \bar{f}_i] - \boldsymbol{L}_{\bar{f}_i} \alpha e_n = \beta[e_{n-1}, \bar{f}_i] - \boldsymbol{L}_{\bar{f}_i} \beta e_{n-1}$$

$$\in span\{e_n, e_{n-1}\} = span\{\bar{g}_0, ad_{\bar{f}_0} \bar{g}_0\} = \bar{G}^1$$

$$\vdots$$

$$[ad_{\bar{f}_0}^{n-2} \bar{g}_0, \bar{f}_i] \in \bar{G}^{n-2}$$

Theorem: the question is possible iff $\dot{\xi} = f_0(\xi) + g_0(\xi)u$ is feedback linearisable and

$$[ad_{f_0}^j g_0, f_i] \in G^j \quad 0 \le j \le n-2, \ 1 \le i \le p$$

4.2 Partial Parametric Strict-Feedback Canonical Forms

Let

$$\dot{\xi} = f_0(\xi) + \sum_{i=1}^{p} \theta_i f_i(\xi) + g_0(\xi) u$$

Suppose $\exists\ \lambda$ such that $y = \lambda(\xi)$ with this system has relative degree ρ when $\theta = 0$. Then $\exists$ diffeomorphism $x = \Phi(\xi)$ such that

$$\dot{x}_1 = x_2 + \psi_1^T(x)\theta$$

$$\dot{x}_2 = x_3 + \psi_2^T(x)\theta$$

$$\vdots$$

$$\dot{x}_{\rho-1} = x_\rho + \psi_{\rho-1}^T(x)\theta$$

$$\dot{x}_\rho = \psi_{\rho 0}(x) + \sigma(x)u + \psi_\rho^T(x)\theta$$

$$\dot{x}_{\rho+1} = \psi_{(\rho+1)0} + \psi_{\rho+1}^T(x)\theta$$

$$\vdots$$

$$\dot{x}_n = \psi_{n0} + \psi_n^T(x)\theta$$

$$y = x_1$$

We want ψ_i depends only on $x_1, \cdots, x_i$, $i = 1, \cdots, \rho - 1$

As before, try

$$[ad_{f_0}^j g_0, f_i] \in G^j, \quad j = 0, \cdots, \rho - 2$$

$$\Downarrow$$

$j = 0$: $G^0 = span\{g_0\} = span\{e_\rho\}$

$\Rightarrow [g_0, f_i] \in span\{e_\rho\} \Rightarrow \left[\sigma e_\rho, f_i\right] \in span\{e_\rho\}$

$f_i = \sum_{j=1}^n \psi_{ji} e_j$

$\Rightarrow [\sigma e_\rho, f_i] = \sigma \sum_{j=1}^n \frac{\partial \psi_{ji}}{\partial e_\rho} e_j - \boldsymbol{L}_{f_i} \sigma e_\rho \Rightarrow \frac{\partial \psi_{ji}}{\partial x_\rho} = 0$

$\Rightarrow \psi_i(x) = \psi_i(x_1, \cdots, x_{\rho-1}, x_{\rho+1}, \cdots, x_n)$

We want ψ_i to be independent of $x_{\rho+1}, \cdots, x_n$ for $i = 1, \cdots, \rho$ and don't care about ψ_i being independent of x_ρ for $i = \rho+1, \cdots, n$.

$\Rightarrow$ No good $\Rightarrow$ Try something else

Note that

$$\bar{f}_i = \left[\frac{d\Phi}{d\xi} f_i\right]_{\xi=\Phi^{-1}(x)}$$

$$\Downarrow$$

$$\psi_{ji} = [d(L_{f_0}^{j-1}\lambda) f_i] \circ \Phi^{-1} = (L_{f_i} L_{f_0}^{j-1}\lambda) \circ \Phi^{-1}$$

Hence,

$$\begin{bmatrix} \psi_{1i} \\ \psi_{2i} \\ \vdots \\ \psi_{\rho i} \end{bmatrix} = \begin{bmatrix} \psi_{1i}(x_1) \\ \psi_{2i}(x_1, x_2) \\ \vdots \\ \psi_{\rho i}(x_1, \cdots, x_\rho) \end{bmatrix}$$

$$\Updownarrow$$

$$d\psi_{1i} \in span\{dx_1\}$$

$$d\psi_{2i} \in span\{dx_1, dx_2\}$$

$$\vdots$$

$$d\psi_{\rho i} \in span\{dx_1, \cdots, dx_\rho\}$$

$$\Updownarrow$$

$$d(L_{\bar{f}_i}\bar{\lambda}) = d(d\bar{\lambda}\bar{f}_i) = d\left\{[d\lambda]_{\xi=\Phi^{-1}(x)} \frac{d(\Phi^{-1})}{dx} \left[\left(\frac{d\Phi}{d\xi}\right) f_i\right]_{\xi=\Phi^{-1}(x)}\right\}$$

$$= d\left([d\lambda f_i]_{\xi=\Phi^{-1}(x)}\right) = d\left([L_{f_i}\lambda]_{\xi=\Phi^{-1}(x)}\right) = d\psi_{1i} \in span\{dx_1\} = span\{d\bar{\lambda}\}$$

$$d(L_{\bar{f}_i} L_{\bar{f}_0}\bar{\lambda}) = d([L_{f_i} L_{f_0}\lambda]_{\xi=\Phi^{-1}(x)}) = d\psi_{2i} \in span\{dx_1, dx_2\} = span\{d\bar{\lambda}, d(L_{\bar{f}_0}\bar{\lambda})\}$$

$$\vdots$$

$$d(L_{\bar{f}_i} L_{\bar{f}_0}^{\rho-2}\bar{\lambda}) \in span\{d\bar{\lambda}, \cdots, d(L_{\bar{f}_0}^{\rho-2}\bar{\lambda})\}$$

where

$$\bar{\lambda}(x) = \lambda(\Phi^{-1}(x)) = x_1$$

Hence, under this condition, the transformed system will have the form

$$\dot{x}_1 = x_2 + \psi_1^T(x_1)\theta$$

$$
\begin{aligned}
\dot{x}_2 &= x_3 + \psi_2^T(x_1, x_2)\theta \\
&\vdots \\
\dot{x}_{\rho-1} &= x_n + \psi_{\rho-1}^T(x_1, \cdots, x_{\rho-1})\theta \\
\dot{x}_\rho &= \psi_{\rho 0}(x) + \sigma(x)u + \psi_\rho^T(x)\theta \\
\dot{x}_{\rho+1} &= \psi_{(\rho+1)0} + \psi_{\rho+1}^T(x)\theta \\
&\vdots \\
\dot{x}_n &= \psi_{n0} + \psi_n^T(x)\theta
\end{aligned}
$$

Chapter 5

Stabilization by Backstepping

In this chapter, we will consider the global asymptotic stabilization of a class of linear in input systems by the technique called "Backstepping". Before we proceed to the main results, we first explore the global asymptotic stabilizability of the system.

Def: A smooth positive definite and radially unbounded function $V : \mathbf{R}^n \to \mathbf{R}^+$ is called a control Lyapunov function if

$$\inf_{u \in \mathbf{R}} \left\{ \frac{dV}{dx}(x)[f(x) + g(x)u] \right\} < 0 \qquad \forall x \neq 0.$$

Theorem: If $\exists$ a control Lyapunov function, then we can find $u = \alpha(x)$ such that

$$\dot{V} = \frac{dV}{dx}(x)[f(x) + g(x)\alpha(x)] \leq -W$$

for some positive definite W and, thus, the closed loop system is globally asymptotically stable.

Throughout this chapter, the existence of control Lyapunov function is assumed.

5.1 Integrator Backstepping

Consider

$$\begin{aligned} \dot{x} &= \cos x - x^3 + \xi \\ \dot{\xi} &= u. \end{aligned}$$

Want $x(t) \to 0$ as $t \to \infty$.

Only equil. pt with $x = 0$: $(0, -1) \Rightarrow$ Want $(0, -1)$ globally asymptotically stable.

Consider the x-subsystem where ξ is the control input. Then the feedback linearising control + pole placement at -1,

$$\xi = -\cos x + x^3 - x$$

yields

$$\dot{x} = -x.$$

Let $V(x) = \frac{1}{2}x^2$.

Then $\dot{V}(x) \leq -x^2 \Rightarrow \inf_\xi \frac{dV}{dx}(f(x) + g(x)\xi) = \inf_\xi x(\cos x - x^3 + \xi) \leq -x^2 < 0 \Rightarrow V$ is a control Lyapunov function.

Another simpler stabilising controller is $\xi = -\cos x - c_1 x$. Clearly, a Lyapunov function for the closed loop system is $V(x) = \frac{1}{2}x^2$.

We now return to the full system. Consider ξ as a virtual control to the x-subsystem. Then the desired ξ is

$$\xi_{des} = -c_1 x - \cos x =: \alpha(x).$$

Define the error variable

$$z = \xi - \xi_{des} = \xi - \alpha(x) = \xi + c_1 x + \cos x$$

$$\Downarrow$$

$$\dot{x} = \cos x - x^3 + [\xi + c_1 x + \cos x] - c_1 x - \cos x = -c_1 x - x^3 + z$$

$$\dot{z} = \dot{\xi} - \dot{\alpha} = \dot{\xi} + (c_1 - \sin x)\dot{x} = u + (c_1 - \sin x)\left(-c_1 x - x^3 + z\right).$$

Try

$$V_a(x, \xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + c_1 x + \cos x)^2$$

$$\Downarrow$$

$$\begin{aligned}\dot{V}_a(x, z, u) &= x[-c_1 x - x^3 + z] + z\left[u + (c_1 - \sin x)\left(-c_1 x - x^3 + z\right)\right] \\ &= -c_1 x^2 - x^4 + z\left[x + u + (c_1 - \sin x)\left(-c_1 x - x^3 + z\right)\right].\end{aligned}$$

To make the stuff inside the bracket $-c_2 z^2$ with $c_2 > 0$ for the negative definiteness of $\dot{V}_a$, pick

$$u = -c_2 z - x - (c_1 - \sin x)\left(-c_1 x - x^3 + z\right)$$

$$\Downarrow$$

$$\dot{V}_a = -c_1 x^2 - x^4 - c_2 z^2.$$

$\Rightarrow$ Closed loop system is globally asymptotically stable.

Lemma (Integrator Backstepping): Consider

$$\dot{x} = f(x) + g(x)\xi \quad (*)$$

$$\dot{\xi} = u.$$

If $\exists$ a control Lyapunov function V for the system $(*)$ with ξ as its control and $\xi = \alpha(x)$ is the stabilising control for $(*)$,

$$V_a(x, \xi) = V(x) + \frac{1}{2}[\xi - \alpha(x)]^2$$

is the control Lyapunov function for the full system. A globally asymptotically stabilising controller for the full system is

$$u = -c(\xi - \alpha(x)) + \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi] - \frac{dV}{dx}(x)g(x), \quad c > 0.$$

Proof:

$$z := \xi - \alpha(x)$$

$$\Downarrow$$

$$\dot{x} = f(x) + g(x)[\alpha(x) + z]$$

$$\dot{z} = u - \frac{d\alpha}{dx}(x)[f(x) + g(x)(\alpha(x) + z)]$$

$$\Downarrow$$

$$\dot{V}_a = \frac{dV}{dx}(f + g\alpha + gz) + z\left[u - \frac{d\alpha}{dx}(f + g(\alpha + z))\right]$$

$$= \frac{dV}{dx}(f + g\alpha) + z\left[u - \frac{d\alpha}{dx}(f + g(\alpha + z)) + \frac{dV}{dx}g\right]$$

$$\leq -W(x) + z\left[u - \frac{d\alpha}{dx}(f + g(\alpha + z)) + \frac{dV}{dx}g\right].$$

Pick

$$u = -c(\xi - \alpha(x)) + \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi] - \frac{dV}{dx}(x)g(x), \quad c > 0$$

$$\Downarrow$$

$$\dot{V}_a \leq -W(x) - cz^2 < 0 \quad \text{if} \quad (x,z) \neq 0.$$

5.2 Backstepping for Strict Feedback Systems

Consider the strict feedback system:

$$\dot{x} = f(x) + g(x)\xi_1$$

$$\dot{\xi}_1 = f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2$$
$$\dot{\xi}_2 = f_2(x,\xi_1,\xi_2) + g_2(x,\xi_1,\xi_2)\xi_3$$
$$\vdots$$
$$\dot{\xi}_{k-1} = f_{k-1}(x,\xi_1,\cdots,\xi_{k-1}) + g_{k-1}(x,\xi_1,\cdots,\xi_{k-1})\xi_k$$
$$\dot{\xi}_k = f_k(x,\xi_1,\cdots,\xi_k) + g_k(x,\xi_1,\cdots,\xi_k)u$$

where $\exists$ a control Lyapunov function for the x-subsystem.

First consider the subsystem

$$\dot{x} = f(x) + g(x)\xi_1$$

$$\dot{\xi}_1 = f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2$$

where ξ_2 is virtual control. Try

$$V_1(x,\xi_1) = V(x) + \frac{1}{2}[\xi_1 - \alpha(x)]^2$$

Need $\xi_2 = \alpha_1$ such that $\dot{V}_1$ is negative definite.

$$\begin{aligned}\dot{V}_1 &\le -W(x) + [\xi_1 - \alpha(x)]\left\{\frac{dV}{dx}(x)g(x) + f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \right.\\ &\qquad \left. - \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi_1]\right\}\\ &= -W(x) + [\xi_1 - \alpha(x)]\left\{\frac{dV}{dx}(x)g(x) + f_1(x,\xi_1) + g_1(x,\xi_1)\alpha_1(x,\xi_1)\right.\\ &\qquad \left. + g_1(x,\xi_1)[\xi_2 - \alpha_1(x,\xi_1)] - \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi_1]\right\}\\ &= -W_1(x,\xi_1) + \frac{dV_1}{d\xi_1}(x,\xi_1)g_1(x,\xi_1)[\xi_2 - \alpha_1(x,\xi_1)].\end{aligned}$$

Pick α_1 such that $W_1(x,\xi_1) > 0$ if $(x,\xi_1) \neq (0,\alpha(x))$.

If $g_1 \neq 0$ for all (x,ξ_1), one possible choice is

$$\begin{aligned}\alpha_1(x,\xi_1) := \frac{1}{g_1(x,\xi_1)}&\left\{-c_1[\xi_1 - \alpha(x)] - \frac{dV}{dx}(x)g(x) - f_1(x,\xi_1)\right.\\ &\left. + \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi_1]\right\}.\end{aligned}$$

Then $W_1(x,\xi_1) = W(x) + c_1[\xi_1 - \alpha(x)]^2$.

With appropriate α_1, consider

$$\dot{X}_1 = F_1(X_1) + G_1(X_1)\xi_2$$

$$\dot{\xi}_2 = f_2(X_1,\xi_2) + g_2(X_1,\xi_2)\xi_3$$

where

$$X_1 = \begin{bmatrix} x \\ \xi_1 \end{bmatrix}, \quad F_1(X_1) = \begin{bmatrix} f(x) + g(x)\xi_1 \\ f_1(x,\xi_1) \end{bmatrix}, \quad G_1(X_1) = \begin{bmatrix} 0 \\ g_1(x,\xi_1) \end{bmatrix}.$$

Try

$$V_2(X_1,\xi_2) = V_1(X_1) + \frac{1}{2}[\xi_2 - \alpha_1(X_1)]^2$$

$$= V(x) + \frac{1}{2}\sum_{i=1}^{2}[\xi_i - \alpha_{i-1}(\boldsymbol{X}_{i-1})]^2.$$

Then

$$\dot{V}_2 \le -\boldsymbol{W}_2(\boldsymbol{X}_1, \xi_2) + \frac{d\boldsymbol{V}_2}{d\xi_2}(\boldsymbol{X}_1, \xi_2) g_1(\boldsymbol{X}_1, \xi_2)[\xi_3 - \alpha_2(\boldsymbol{X}_2)]$$

and, for some appropriate α_2, $\boldsymbol{W}_2(\boldsymbol{X}_1, \xi_2) > 0$ if $(\boldsymbol{X}_1, \xi_1) \neq (0, \alpha(x), \alpha_1(\boldsymbol{X}_1))$.

$$\vdots$$

At the kth step, consider

$$\dot{\boldsymbol{X}}_{k-1} = \boldsymbol{F}_{k-1}(\boldsymbol{X}_{k-1}) + \boldsymbol{G}_{k-1}(\boldsymbol{X}_{k-1})\xi_k$$

$$\dot{\xi}_k = f_k(\boldsymbol{X}_{k-1}, \xi_k) + g_k(\boldsymbol{X}_{k-1}, \xi_k)u$$

where

$$\boldsymbol{X}_{k-1} = \begin{bmatrix} \boldsymbol{X}_{k-2} \\ \xi_{k-1} \end{bmatrix}, \quad \boldsymbol{F}_{k-1}(\boldsymbol{X}_{k-1}) = \begin{bmatrix} \boldsymbol{F}_{k-2}(\boldsymbol{X}_{k-2}) + \boldsymbol{G}_{k-2}(\boldsymbol{X}_{k-2})\xi_{k-1} \\ f_{k-1}(\boldsymbol{X}_{k-2}, \xi_{k-1}) \end{bmatrix},$$

$$\boldsymbol{G}_{k-1}(\boldsymbol{X}_{k-1}) = \begin{bmatrix} 0 \\ g_{k-1}(\boldsymbol{X}_{k-2}, \xi_{k-1}) \end{bmatrix}.$$

Try

$$V_k(x, \xi_1, \cdots, \xi_k) = V_{k-1}(\boldsymbol{X}_{k-1}) + \frac{1}{2}[\xi_k - \alpha_{k-1}(\boldsymbol{X}_{k-1})]^2$$

$$= V(x) + \frac{1}{2}\sum_{i=1}^{k}[\xi_i - \alpha_{i-1}(\boldsymbol{X}_{i-1})]^2.$$

Then

$$\dot{V}_k = \dot{V}_{k-1} + (\xi_k - \alpha_{k-1})\left[f_k + g_k u - \frac{d\alpha_{k-1}}{d\boldsymbol{X}_{k-1}}(\boldsymbol{F}_{k-1} + \boldsymbol{G}_{k-1}\xi_k)\right]$$

$$\le -\boldsymbol{W}_{k-1}(\boldsymbol{X}_{k-2}, \xi_{k-1}) + \frac{d\boldsymbol{V}_{k-1}}{d\xi_{k-1}} g_{k-1}(\xi_k - \alpha_{k-1})$$

$$+(\xi_k - \alpha_{k-1})\left[f_k + g_k u - \frac{d\alpha_{k-1}}{d\boldsymbol{X}_{k-1}}[\boldsymbol{F}_{k-1} + \boldsymbol{G}_{k-1}\xi_k]\right]$$

$$
\begin{aligned}
&= -W_{k-1}(X_{k-2},\xi_{k-1}) + (\xi_k - \alpha_{k-1})\left[\frac{dV_{k-1}}{d\xi_{k-1}} g_{k-1} + f_k + g_k u\right.\\
&\qquad \left. - \frac{d\alpha_{k-1}}{dX_{k-1}}(F_{k-1} + G_{k-1}\xi_k)\right]\\
&\leq -W_k(X_{k-1},\xi_k) \leq 0.
\end{aligned}
$$

Pick u such that $W_k(x_{k-1},\xi_k) > 0$ if $(x_{k-1},\xi_k) \neq (0,\alpha(x),\alpha_1(x),\cdots,\alpha_{k-1}(x))$.

If $g_k \neq 0$ for all $(x,\xi_1,\cdots,\xi_k)$, one possible choice is

$$
u = \frac{1}{g_k}\left[-c_k(\xi_k - \alpha_{k-1}) - \frac{dV_{k-1}}{d\xi_{k-1}} g_{k-1} - f_k + \frac{d\alpha_{k-1}}{dX_{k-1}}(F_{k-1} + G_{k-1}\xi_k)\right].
$$

Then $W_k = W_{k-1} + c_k[\xi_k - \alpha_{k-1}]^2$.

5.3 Robust Backstepping

5.3.1 Nonlinear Damping for Systems with Matched Uncertainty

Nonlinear system with matched uncertainty:

$$
\dot{x} = u + \psi(x)\Delta(t)
$$

where $\psi(x)$ is known and $\Delta(t)$ is bounded.

Case I: $\Delta(t) = \Delta(0)e^{-kt}$

Try $u = -cx$. Then

$$
\dot{x} = -cx + \psi(x)\Delta(0)e^{-kt}.
$$

Suppose $\psi(x) = x^2$.

$$
\dot{x} = -cx + x^2\Delta(0)e^{-kt}.
$$

Change of variable: $w = \frac{1}{x}$

$$
\dot{w} = -\frac{1}{x^2}\dot{x} = c\frac{1}{x} - \Delta(0)e^{-kt} = cw - \Delta(0)e^{-kt}
$$

$$
\Downarrow
$$

$$w(t) = \left[w(0) - \frac{\Delta(0)}{c+k}\right] e^{ct} + \frac{\Delta(0)}{c+k} e^{-kt}$$
$$\Downarrow$$
$$x(t) = \frac{x(0)(c+k)}{[c+k-\Delta(0)x(0)]e^{ct} + \Delta(0)x(0)e^{-kt}}.$$

The denominator is zero at time t_f if

$$[c+k-\Delta(0)x(0)]e^{ct_f} = -\Delta(0)x(0)e^{-kt_f}$$
$$\Downarrow$$
$$e^{(c+k)t_f} = \frac{\Delta(0)x(0)}{\Delta(0)x(0)-(c+k)}$$
$$\Downarrow$$
$$t_f = \frac{1}{(c+k)} \ln \left[\frac{\Delta(0)x(0)}{\Delta(0)x(0)-(c+k)}\right].$$

To recap, $\Delta(0)x(0) > c+k > 0 \Rightarrow$

$$x(t) \to \infty \quad \text{as} \quad t \to t_f = \frac{1}{(c+k)} \ln \left[\frac{\Delta(0)x(0)}{\Delta(0)x(0)-(c+k)}\right]$$

$\Rightarrow$ Finite Escape Time.

How can we prevent finite escape time?

Let's try adding a nonlinear damping term in the above control:

$$u = -cx - s(x)x.$$

Design $s(x)$ using $V(x) = \frac{1}{2}x^2$

$$\dot{V} = xu + x\psi(x)\Delta(t) = -cx^2 - x^2 s(x) + x\psi(x)\Delta(t).$$

Pick

$$s(x) = \kappa\psi^2(x), \qquad \kappa > 0$$
$$\Downarrow$$
$$u = -cx - \kappa x\psi^2(x)$$
$$\Downarrow$$

$$\dot{V} = -cx^2 - \kappa x^2\psi^2(x) + x\psi(x)\Delta(t)$$

$$= -cx^2 - \kappa\left[x\psi(x) - \frac{\Delta(t)}{2\kappa}\right]^2 + \frac{\Delta^2(t)}{4\kappa} \leq -cx^2 + \frac{\Delta^2(t)}{4\kappa}$$

$$\Downarrow$$

$$\dot{V} < 0 \quad \text{if} \quad x \notin \mathcal{R} := \left\{x : |x| \leq \frac{||\Delta||_\infty}{2\sqrt{\kappa c}}\right\}$$

$$\Downarrow$$

$$||x||_\infty \leq \max\left\{|x(0)|, \frac{||\Delta||_\infty}{2\sqrt{\kappa c}}\right\}.$$

$$\lim_{t\to\infty} dist\{x(t), \mathcal{R}\} = 0.$$

Lemma: Consider

$$\dot{x} = f(x) + g(x)[u + \psi(x)^T\Delta(x, u, t)]$$

where $\psi(x)$ is $p \times 1$ vector valued smooth function and $\Delta(x, u, t)$ is $p \times 1$ vector valued uniformly bounded function.

If $\exists$ control Lyapunov function for the system without uncertainty,

$$u = \alpha(x) - \kappa\frac{dV}{dx}(x)g(x)|\psi(x)|^2, \qquad \kappa > 0$$

achieves global uniform boundedness of $x(t)$ and convergence to the residual set

$$\mathcal{R} := \left\{x : |x| \leq \gamma^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{||\Delta||_\infty^2}{4\kappa}\right)\right\}$$

where $\gamma_1, \gamma_2, \gamma_3$ are class K_∞ functions such that

$$\gamma_1(|x|) \leq V(x) \leq \gamma_2(|x|)$$

$$\gamma_3(|x|) \leq W(x).$$

5.3.2 Backstepping for Systems with Unmatched Uncertainty

Ex:

$$\dot{x} = \xi + x^2 \arctan \xi \Delta_0(t)$$
$$\dot{\xi} = (1+\xi^2)u + e^{x\xi}\Delta_0(t).$$

Step 1:

$$x^2 \arctan \xi \Delta_0(t) := x^2 \Delta_1(\xi, t) = \psi_1(x)\Delta_1(\xi, t).$$

$$||\Delta_1(\xi,t)||_\infty = ||\Delta_0 \arctan \xi||_\infty = \frac{\pi}{2}||\Delta_0||_\infty.$$

Warning: If we had ξ instead of $\arctan \xi$, we would be in trouble.

$$\alpha_1(x) = -c_1 x - \kappa_1 x \psi_1^2(x).$$

Error Variable: $z = \xi - \alpha_1(x)$

$$\dot{x} = -c_1 x + z - \kappa_1 x \psi_1^2(x) + \psi_1(x)\Delta_1(\xi, t).$$

For $V = \frac{1}{2}x^2$,

$$\dot{V} = zx - c_1 x^2 - \kappa_1 x^2 \psi_1^2 + x\psi_1 \Delta_1$$
$$\leq zx - c_1 x^2 - \kappa_1 x^2 \psi_1^2 + |x\psi_1| ||\Delta_1||_\infty \leq zx - c_1 x^2 + \frac{||\Delta_1||_\infty^2}{4\kappa_1}.$$

Step 2:

$$\dot{x} = -c_1 x + z - \kappa_1 x \psi_1^2(x) + \psi_1(x)\Delta_1(\xi, t)$$
$$\dot{z} = (1+\xi^2)u + e^{x\xi}\Delta_0(t) - \frac{d\alpha_1}{dx}[\xi + x^2 \arctan \xi \Delta_0(t)]$$
$$= (1+\xi^2)u - \frac{d\alpha_1}{dx}\xi + \left[e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan \xi\right]\Delta_0(t)$$

where

$$\frac{d\alpha_1}{dx} = -c_1 - \kappa_1 \frac{d}{dx}[x\psi_1^2(x)] = -c_1 - 5\kappa_1 x^4.$$

Pick

$$u = \frac{1}{1+\xi^2}\left\{-c_2 z + \frac{d\alpha_1}{dx}\xi - x - \kappa_2 z \left[e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan \xi\right]^2\right\}.$$

$$V_2(x,\xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}[\xi - \alpha_1(x)]^2$$

$$\Downarrow$$

$$\dot{V}_2 = \dot{V} + z\dot{z} \le zx - c_1 x^2 + \frac{||\Delta_1||_\infty^2}{4\kappa_1} + z\dot{z}$$

$$= -c_1 x^2 + \frac{||\Delta_1||_\infty^2}{4\kappa_1} + z\left\{-c_2 z - \kappa_2 z\left[e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan\xi\right]^2 \right.$$

$$\left. + \left[e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan\xi\right]\Delta_0(t)\right\}$$

$$\le -c_1 x^2 - c_2 z^2 + \frac{||\Delta_1||_\infty^2}{4\kappa_1} - \kappa_2 z^2\left[e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan\xi\right]^2$$

$$+|z|\left|e^{x\xi} - \frac{d\alpha_1}{dx}x^2 \arctan\xi\right| ||\Delta_0||_\infty$$

$$\le -c_1 x^2 - c_2 z^2 + \frac{||\Delta_1||_\infty^2}{4\kappa_1} + \frac{||\Delta_0||_\infty^2}{4\kappa_2}.$$

$\Rightarrow$ x, z bounded
$\Rightarrow$ $\xi = z + \alpha_1(x)$ bounded
Lemma: Consider

$$\dot{x} = f(x) + g(x)u + F(x)\Delta_1(x,u,t)$$

where $F(x)$ is a known $n \times q$ smooth matrix valued functions and Δ_1 is q vector valued uniformly bounded uncertainty.

Suppose $\exists$ $u = \alpha(x)$ such that

$$\frac{dV}{dx}(x)[f(x) + g(x)\alpha(x) + F(x)\Delta_1(x,u,t)] \le -W(x) + b$$

and thus $x(t)$ is globally bounded.

Now consider

$$\dot{x} = f(x) + g(x)\xi + F(x)\Delta_1(x,u,t)$$

$$\dot{\xi} = u + \psi(x,\xi)^T \Delta_2(x,\xi,u,t)$$

where $\psi(x,\xi)$ is a known $p \times 1$ smooth vector valued functions and Δ_2 is p vector valued uniformly bounded uncertainty. For this system,

$$u = -c[\xi - \alpha(x)] + \frac{d\alpha}{dx}(x)[f(x) + g(x)\xi] - \frac{dV}{dx}(x)g(x)$$

$$-\kappa[\xi - \alpha(x)]\left\{|\psi(x,\xi)|^2 + \left|\frac{d\alpha}{dx}(x)\boldsymbol{F}(x)\right|^2\right\}$$

guarantees global uniform boundedness of $x(t)$ and $\xi(t)$ with $c, \kappa > 0$.

5.3.3 Robust Backstepping for Robust Strict Feedback Systems

$$\dot{x}_1 = x_2 + \phi_1^T(x_1)\Delta(x,u,t)$$
$$\dot{x}_2 = x_3 + \phi_2^T(x_1,x_2)\Delta(x,u,t)$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \phi_{n-1}^T(x_1,\cdots,x_{n-1})\Delta(x,u,t)$$
$$\dot{x}_n = \beta(x)u + \phi_n^T(x)\Delta(x,u,t)$$

Corollary: $x(t)$ is globally uniformly bounded if

$$u = \frac{1}{\beta(x)}\alpha_n(x)$$

where

$$z_i = x_i - \alpha_{i-1}(x_1,\cdots,x_{i-1})$$

$$\alpha_i(x_1,\cdots,x_i) = -c_i z_i - z_{i-1} + \sum_{j=1}^{i-1}\frac{d\alpha_{i-1}}{dx_j}x_{j+1} - \kappa_i z_i\left|\psi_i - \sum_{j=1}^{i-1}\frac{d\alpha_{i-1}}{dx_j}\psi_j\right|^2$$

with $c_i, \kappa_i > 0$.

Chapter 6

Stabilization of Systems with Unknown Parameters by Adaptive Backstepping

6.1 Adaptive Control

$$\dot{x} = u + \theta\psi(x)$$

where θ is unknown constant parameter.

Pick

$$u = -cx - \kappa x\psi^2(x)$$

$$\Downarrow$$

$$\dot{x} = -cx - \kappa x\psi^2(x) + \theta\psi(x).$$

$V = \frac{1}{2}x^2 \Rightarrow$

$$\dot{V} \leq -cx^2 + \frac{\theta^2}{4\kappa}$$

$$\Downarrow$$

$$x(t) \to \left\{x : |x| \leq \frac{|\theta|}{2\sqrt{\kappa c}}\right\}.$$

Hence, the above control guarantees the global boundedness.

Goal: Want $x(t) \to 0$ as $t \to \infty$.

When θ is known, an asymptotically stabilising control law is

$$u = -\theta\psi(x) - c_1 x, \quad c_1 > 0.$$

When θ is unknown, use certainty Equivalence:

$$u = -\hat{\theta}\psi(x) - c_1 x, \quad c_1 > 0$$

where $\hat{\theta}$ is an estimate of θ.

Question: Find the update law for $\hat{\theta}$ that leads to asymptotically stable closed loop system.

Closed loop:

$$\dot{x} = -c_1 x + \tilde{\theta}\psi(x)$$

where the estimation error $\tilde{\theta}$ is defined by

$$\tilde{\theta} := \theta - \hat{\theta}.$$

$V_0(x) = \frac{1}{2}x^2 \Rightarrow \dot{V}_0 = -c_1 x^2 + \tilde{\theta} x\psi(x).$

$$V_1(x, \tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2$$

$$\Downarrow$$

$$\dot{V}_1 = x\dot{x} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} = -c_1 x^2 + \tilde{\theta} x\psi(x) + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} = -c_1 x^2 + \tilde{\theta}\left[x\psi(x) + \frac{1}{\gamma}\dot{\tilde{\theta}}\right].$$

Pick the parameter estimator as

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \gamma x\psi(x)$$

$$\Downarrow$$

$$\dot{V}_1 = -c_1 x^2 \leq 0.$$

Hence, with the adaptive nonlinear control

$$u = -c_1 x - \hat{\theta}\psi(x)$$

$$\dot{\hat{\theta}} = \gamma x\psi(x),$$

the closed loop system is

$$\dot{x} = -c_1 x + \tilde{\theta}\psi(x)$$

$$\dot{\tilde{\theta}} = -\gamma x \psi(x).$$

The equil. pt $x = 0, \tilde{\theta} = 0$ is globally stable.

$\Rightarrow$ $x, \tilde{\theta}$ bounded $\Rightarrow$ $\dot{\tilde{\theta}}$ bounded.

Let $M = \{(x, \tilde{\theta}) : \dot{V}_1 = 0\} = \{(0, \tilde{\theta})\}$. Then

LaSalle's Theorem $\Rightarrow$ $x(t) \to 0$ as $t \to \infty$.

$\dot{V}_1 \equiv 0 \Rightarrow x \equiv 0 \Rightarrow \dot{x} \equiv 0 \Rightarrow \tilde{\theta}\psi(0) \equiv 0$.

If $\psi(0) \neq 0$, then $\tilde{\theta}(t) \to 0$ as $t \to \infty$ and we have global asymptotic stability.

If $\psi(0) = 0$, we don't have GAS because every $(0, \tilde{\theta})$ is an equil pt.

However, $\dot{\tilde{\theta}}(t) \to 0$ as $t \to \infty$.

In general, this doesn't imply $\tilde{\theta}(t) \to$ *const* as $t \to \infty$ (Ex: $\tilde{\theta}(t) = \sin(\ln t)$).

But we can show that $\tilde{\theta}(t) \to$ *const* as $t \to \infty$.

To see this, observe $V_1 = \frac{1}{2}\left(x^2 + \frac{1}{\gamma}\tilde{\theta}^2\right) \geq 0$ and $\dot{V}_1 \leq 0$.

$\Rightarrow$ $\exists V_1(\infty)$ such that $V_1(t) \to V_1(\infty)$ as $t \to \infty$.

$\Rightarrow$ $x^2 + \frac{1}{\gamma}\tilde{\theta}^2 \to 2V_1(\infty)$ as $t \to \infty$.

$\Rightarrow$ $\tilde{\theta}(t) \to \pm\sqrt{2\gamma V_1(\infty)}$ as $t \to \infty$.

6.2 Adaptive Integrator Backstepping

$$\dot{x}_1 = x_2 + \theta\psi(x_1)$$

$$\dot{x}_2 = u.$$

If θ is known,

$$\alpha_1(x_1, \theta) = -c_1 x_1 - \theta\psi(x_1).$$

$$V_c(x, \theta) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - \alpha_1(x_1, \theta))^2.$$

Pick the integrator backstepping controller:

$$u = -c_2(x_2 - \alpha_1(x_1, \theta)) - x_1 + \frac{d\alpha_1}{dx_1}(x_2 + \theta\psi)$$

$$\Downarrow$$

$$\dot{V}_c(x,\theta) = -c_1 x_1^2 - c_2(x_2 - \alpha_1(x_1,\theta))^2.$$

If θ is unknown,

Step 1: x_2 virtual control $\Rightarrow$

$$\alpha_1(x_1,\vartheta_1) = -c_1 x_1 - \vartheta_1 \psi(x_1)$$

$$\dot{\vartheta}_1 = \gamma x_1 \psi(x_1).$$

Error variable: $z_2 := x_2 - \alpha_1(x_1,\vartheta_1)$

For uniformity: $z_1 := x_1$

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1)\psi.$$

$$V_1(x,\vartheta_1) = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}(\theta - \vartheta_1)^2$$

$$\Downarrow$$

$$\dot{V}_1 = z_1\dot{z}_1 - \frac{1}{\gamma}(\theta - \vartheta_1)\dot{\vartheta}_1 = z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1)\left(\psi z_1 - \frac{1}{\gamma}\dot{\vartheta}_1\right) = z_1 z_2 - c_1 z_1^2.$$

Step 2:

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u - \frac{d\alpha_1}{dx_1}\dot{x}_1 - \frac{d\alpha_1}{d\vartheta_1}\dot{\vartheta}_1 = u - \frac{d\alpha_1}{dx_1}(x_2 + \theta\psi) - \frac{d\alpha_1}{d\vartheta_1}\gamma\psi z_1$$

$$= u - \frac{d\alpha_1}{dx_1}x_2 - \frac{d\alpha_1}{d\vartheta_1}\gamma\psi z_1 - \theta\frac{d\alpha_1}{dx_1}\psi.$$

$$V_2(z_1,z_2,\vartheta_1) = V_1(z_1,\vartheta_1) + \frac{1}{2}z_2^2$$

$$\Downarrow$$

$$\dot{V}_2 = \dot{V}_1 + z_2\dot{z}_2 = -c_1 z_1^2 + z_2\left[z_1 + u - \frac{d\alpha_1}{dx_1}x_2 - \frac{d\alpha_1}{d\vartheta_1}\gamma\psi z_1 - \theta\frac{d\alpha_1}{dx_1}\psi\right].$$

Choose

$$u = -z_1 - c_2 z_2 + \frac{d\alpha_1}{dx_1}x_2 + \frac{d\alpha_1}{d\vartheta_1}\gamma\psi z_1 + \vartheta_1\frac{d\alpha_1}{dx_1}\psi$$

$$\Downarrow$$

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_1)\frac{d\alpha_1}{dx_1}\psi z_2.$$

The RHS may not be ≤ 0.

Want to cancel the last term $\Rightarrow$ Introduce new estimate ϑ_2 (Overparametrisation)

$$u = -z_1 - c_2 z_2 + \frac{d\alpha_1}{dx_1}x_2 + \frac{d\alpha_1}{d\vartheta_1}\gamma\psi z_1 + \vartheta_2\frac{d\alpha_1}{dx_1}\psi$$

$$\Downarrow$$

$$\dot{z}_2 = -c_2 z_2 - z_1 - (\theta - \vartheta_2)\frac{d\alpha_1}{dx_1}\psi.$$

$$V_2(z_1, z_2, \vartheta_1, \vartheta_2) = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(\theta - \vartheta_2)^2$$

$$= \frac{1}{2}(z_1^2 + z_2^2) + \frac{1}{2\gamma}[(\theta - \vartheta_1)^2 + (\theta - \vartheta_2)^2]$$

$$\Downarrow$$

$$\dot{V}_2 = \dot{V}_1 + z_2\dot{z}_2 - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2$$

$$= z_1 z_2 - c_1 z_1^2 + z_2\left[-c_2 z_2 - z_1 - (\theta - \vartheta_2)\frac{d\alpha_1}{dx_1}\psi\right] - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2$$

$$= -c_1 z_1^2 - c_2 z_2^2 - (\theta - \vartheta_2)\left(z_2\frac{d\alpha_1}{dx_1}\psi + \frac{1}{\gamma}\dot{\vartheta}_2\right).$$

Pick

$$\dot{\vartheta}_2 = -\gamma\frac{d\alpha_1}{dx_1}\psi z_2$$

$$\Downarrow$$

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2.$$

The closed loop system:

$$\dot{z}_1 = -c_1 z_1 + z_2 + (\theta - \vartheta_1)\psi$$

$$\dot{z}_2 = -c_2 z_2 - z_1 - (\theta - \vartheta_2)\frac{d\alpha_1}{dx_1}\psi$$

$$\dot{\vartheta}_1 = \gamma\psi z_1$$

$$\dot{\vartheta}_2 = -\gamma \frac{d\alpha_1}{dx}\psi z_2$$

or

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \psi & 0 \\ 0 & -\frac{d\alpha_1}{dx_1}\psi \end{bmatrix}\begin{bmatrix} \theta - \vartheta_1 \\ \theta - \vartheta_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \gamma \begin{bmatrix} \psi & 0 \\ 0 & -\frac{d\alpha_1}{dx_1}\psi \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Assumption: Consider

$$\dot{x} = f(x) + \boldsymbol{F}(x)\theta + g(x)u$$

$\exists$

$$u = \alpha(x, \vartheta)$$
$$\dot{\vartheta} = \boldsymbol{T}(x, \vartheta)$$

and $\boldsymbol{V}(x, \vartheta)$ PD and radially unbounded in $(x, \vartheta - \theta)$ such that

$$\frac{d\boldsymbol{V}}{dx}(x,\vartheta)[f(x) + \boldsymbol{F}(x)\theta + g(x)\alpha(x,\vartheta)] + \frac{d\boldsymbol{V}}{d\vartheta}(x,\vartheta)\boldsymbol{T}(x,\vartheta) \leq -\underbrace{\boldsymbol{W}(x,\vartheta)}_{PD} \leq 0.$$

Lemma:

$$\dot{x} = f(x) + \boldsymbol{F}(x)\theta + g(x)\xi$$
$$\dot{\xi} = u.$$

Consider

$$u = -c(\xi - \alpha(x,\vartheta)) + \frac{d\alpha}{dx}(x,\vartheta)[f(x) + \boldsymbol{F}(x)\vartheta + g(x)\xi]$$
$$+\frac{d\alpha}{d\vartheta}\boldsymbol{T}(x,\vartheta) - \frac{d\boldsymbol{V}}{dx}(x,\vartheta)g(x), \quad c > 0$$
$$\dot{\vartheta} = \boldsymbol{T}(x,\vartheta)$$
$$\dot{\varTheta} = -\Gamma\left[\frac{d\alpha}{dx}(x,\vartheta)\boldsymbol{F}(x)\right]^T (\xi - \alpha(x,\vartheta))$$

where $\varTheta$ is a new estimate of θ, $\Gamma = \Gamma^T$ is the adpatation gain matrix. Under the assumption, the adaptive controller guarantees global boundedness of $x(t), \xi(t), \vartheta(t), \varTheta(t)$ and regulation of $\boldsymbol{W}(x(t), \vartheta(t))$ and $\xi(t) - \alpha(x(t), \vartheta(t))$. the associated Lyapunov function is

$$\boldsymbol{V}_a(x, \xi, \vartheta, \varTheta) = \boldsymbol{V}(x,\vartheta) + \frac{1}{2}[\xi - \alpha(x,\vartheta)]^2 + \frac{1}{2}(\theta - \varTheta)^T\Gamma^{-1}(\theta - \varTheta).$$

6.3 Adaptive Backstepping for Parametric Strict-Feedback Systems

Consider the parametric strict feedback canonical form:

$$\dot{x}_1 = x_2 + \psi_1^T(x_1)\theta$$

$$\dot{x}_2 = x_3 + \psi_2^T(x_1, x_2)\theta$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n + \psi_{n-1}^T(x_1, \cdots, x_{n-1})\theta$$

$$\dot{x}_n = \beta(x)u + \psi_n^T(x)\theta.$$

Theorem: Suppose $\beta(x) \neq 0$ for all $x \in \mathbf{R}^n$. Consider

$$u = \frac{1}{\beta(x)}\alpha_n(x, \vartheta_1, \cdots, \vartheta_n)$$

$$\dot{\vartheta}_i = \Gamma\left(\psi_i - \sum_{j=1}^{i-1}\frac{d\alpha_{i-1}}{dx_j}\psi_j\right)z_i, \quad i = 1, \cdots, n$$

where

$$z_i = x_i - \alpha_{i-1}(x_1, \cdots, x_{i-1}, \vartheta_1, \cdots, \vartheta_{i-1})$$

$$\alpha_i = -c_i z_i - z_{i-1} - \left(\psi_i - \sum_{j=1}^{i-1}\frac{d\alpha_{i-1}}{dx_j}\psi_j\right)^T\vartheta_i$$

$$+\sum_{j=1}^{i-1}\left[\frac{d\alpha_{i-1}}{dx_j}x_{j+1} + \frac{d\alpha_{i-1}}{d\vartheta_j}\Gamma\left(\psi_j - \sum_{k=1}^{j-1}\frac{d\alpha_{j-1}}{dx_k}\psi_k\right)z_j\right].$$

The above controller guarantees global boundedness of $x(t), \vartheta_1(t), \cdots, \vartheta_n(t)$ and regulation of $x_1(t)$ and $x_i(t) - x_i^e$, $i = 2, \cdots, n$, where $x_i^e = -\theta^T\psi_{i-1}(0, x_2^e, \cdots, x_{i-1}^e)$.

6.4 Extended Matching Design

6.4.1 Reducing Overparametrization

$$\dot{x}_1 = x_2 + \theta\psi(x_1)$$
$$\dot{x}_2 = u$$

Goal: Want to avoid overparametrization.
Strategy: leave $\dot{\hat{\theta}}$ undetermined in Step 1.
Step 1:

$$\alpha_1 = -c_1 z_1 - \hat{\theta}\psi$$
$$\Downarrow$$
$$\dot{z}_1 = -c_1 z_1 + z_2 + \tilde{\theta}\psi.$$

$$V_1(z_1, \tilde{\theta}) = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{\theta}^2$$
$$\Downarrow$$
$$\dot{V}_1 = z_1 z_2 - c_1 z_1^2 + \tilde{\theta}\left(\psi z_1 - \frac{1}{\gamma}\dot{\hat{\theta}}\right).$$

Step 2:

$$\dot{z}_2 = u - \frac{d\alpha_1}{dx_1}(x_2 + \theta\psi) - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}} = u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \tilde{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}.$$

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2$$
$$\Downarrow$$
$$\dot{V}_2 = z_1 z_2 - c_1 z_1^2 + \tilde{\theta}\left(\psi z_1 - \frac{1}{\gamma}\dot{\hat{\theta}}\right) + z_2\left[u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \tilde{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}\right]$$
$$= -c_1 z_1^2 + \tilde{\theta}\left[\psi z_1 - z_2\frac{d\alpha_1}{dx_1}\psi - \frac{1}{\gamma}\dot{\hat{\theta}}\right] + z_2\left[z_1 + u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}\right].$$

Choose

$$\dot{\hat{\theta}} = \gamma\left(\psi z_1 - \frac{d\alpha_1}{dx_1}\psi z_2\right)$$

and

$$u = -z_1 - c_2 z_2 + \frac{d\alpha_1}{dx_1} x_2 - \hat{\theta} \frac{d\alpha_1}{dx_1} \psi - \frac{d\alpha_1}{d\hat{\theta}} \dot{\hat{\theta}}$$

$$\Downarrow$$

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \leq 0.$$

The closed loop system:

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \psi \\ -\frac{d\alpha_1}{dx_1} \psi \end{bmatrix} \tilde{\theta}$$

$$\dot{\hat{\theta}} = \gamma \begin{bmatrix} \psi & -\frac{d\alpha_1}{dx_1} \psi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

$\Rightarrow z_1, z_2, \tilde{\theta}$ bounded and $z_1(t), z_2(t) \to 0$ as $t \to \infty$.

If $\psi(0) \neq 0$, $\tilde{\theta} \to 0$ as $t \to \infty$.

6.4.2 Boundedness without Adaptation and Improving Transient Performance

Goals: achieve boundedness without adaptation through nonlinear damping and improve transient performance via trajectory initialization.

For this, consider

$$\dot{x}_1 = x_2 + \theta \psi(x_1)$$
$$\dot{x}_2 = u$$
$$y = x_1.$$

Step 1: Error variable

$$z_1 = y - y_r = x_1 - y_r$$

$$\Downarrow$$

$$\dot{z}_1 = x_2 + \theta \psi(x_1) - \dot{y}_r.$$

$$\alpha_1 = -c_1 z_1 - \kappa_1 z_1 \psi^2 - \hat{\theta} \psi + \dot{y}_r$$

$$\Downarrow$$

$$\dot{z}_1 = -c_1 z_1 - \kappa_1 z_1 \psi^2 + z_2 + \tilde{\theta}\psi, \quad z_2 = x_2 - \alpha_1.$$

$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}\tilde{\theta}^2 \Rightarrow$

$$\dot{V}_1 = z_1 z_2 - c_1 z_1^2 - \kappa_1 z_1^2 \psi^2 + \tilde{\theta}\left(\psi z_1 - \frac{1}{\gamma}\dot{\hat{\theta}}\right).$$

Step 2:

$$\dot{z}_2 = u - \frac{d\alpha_1}{dx_1}(x_2 + \theta\psi) - \frac{d\alpha_1}{dy_r}\dot{y}_r - \frac{d\alpha_1}{d\dot{y}_r}\ddot{y}_r - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}$$

$$= u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \tilde{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{dy_r}\dot{y}_r - \ddot{y}_r - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}.$$

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2$$

$$\Downarrow$$

$$\dot{V}_2 = z_1 z_2 - c_1 z_1^2 - \kappa_1 z_1^2 \psi^2 + \tilde{\theta}\left(\psi z_1 - \frac{1}{\gamma}\dot{\hat{\theta}}\right)$$

$$+z_2\left[u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \tilde{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{dy_r}\dot{y}_r - \ddot{y}_r - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}\right]$$

$$= -c_1 z_1^2 - \kappa_1 z_1^2 \psi^2 + \tilde{\theta}\left[\psi z_1 - z_2\frac{d\alpha_1}{dx_1}\psi - \frac{1}{\gamma}\dot{\hat{\theta}}\right]$$

$$+z_2\left[z_1 + u - \frac{d\alpha_1}{dx_1}x_2 - \hat{\theta}\frac{d\alpha_1}{dx_1}\psi - \frac{d\alpha_1}{dy_r}\dot{y}_r - \ddot{y}_r - \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}\right].$$

Choose

$$\dot{\hat{\theta}} = \gamma\left(\psi z_1 - \frac{d\alpha_1}{dx_1}\psi z_2\right)$$

and

$$u = -z_1 - c_2 z_2 - \kappa_2 z_2\left(\frac{d\alpha_1}{dx_1}\psi\right)^2 + \frac{d\alpha_1}{dx_1}x_2 + \hat{\theta}\frac{d\alpha_1}{dx_1}\psi + \frac{d\alpha_1}{dy_r}\dot{y}_r + \ddot{y}_r + \frac{d\alpha_1}{d\hat{\theta}}\dot{\hat{\theta}}$$

$$\Downarrow$$

$$\dot{V}_2 = -c_1 z_1^2 - \kappa_1 z_1^2 \psi^2 - c_2 z_2^2 - \kappa_2 z_2^2 \left(\frac{d\alpha_1}{dx_1}\psi\right)^2 \le 0.$$

The closed loop system is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 - \kappa_1\psi^2 & 1 \\ -1 & -c_2 - \kappa_2\left(\frac{d\alpha_1}{dx_1}\psi\right)^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \psi \\ -\frac{d\alpha_1}{dx_1}\psi \end{bmatrix}\tilde{\theta}$$

$$\dot{\tilde{\theta}} = \gamma\left[\psi \;\; -\frac{d\alpha_1}{dx_1}\psi\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Global stability and asymptotic tracking

$z = 0, \tilde{\theta} = 0$ is globally uniformly stable and the asymptotic tracking $\lim_{t\to\infty} z(t) = 0$ is achieved.

Boundedness without adaptation

$\gamma = 0 \Rightarrow$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 - \kappa_1\psi^2 & 1 \\ -1 & -c_2 - \kappa_2\left(\frac{d\alpha_1}{dx_1}\psi\right)^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \psi \\ -\frac{d\alpha_1}{dx_1}\psi \end{bmatrix}\tilde{\theta}.$$

$$V = \frac{1}{2}|z|^2 = \frac{1}{2}(z_1^2 + z_2^2)$$

$$\Downarrow$$

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2 - \kappa_1 z_1^2\psi^2 - \kappa_2 z_2^2\left(\frac{d\alpha_1}{dx_1}\psi\right)^2 + z_1\psi\tilde{\theta} - z_2\frac{d\alpha_1}{dx_1}\psi\tilde{\theta}$$

$$= -c_1 z_1^2 - c_2 z_2^2 - \kappa_1\left(z_1\psi - \frac{\tilde{\theta}}{2\kappa_1}\right)^2 + \frac{\tilde{\theta}^2}{4\kappa_1} - \kappa_2\left(z_2\frac{d\alpha_1}{dx_1}\psi + \frac{\tilde{\theta}}{2\kappa_2}\right)^2 + \frac{\tilde{\theta}^2}{4\kappa_2}$$

$$\le -c_1 z_1^2 - c_2 z_2^2 + \frac{\tilde{\theta}^2}{4\kappa_1} + \frac{\tilde{\theta}^2}{4\kappa_2} \le -c_0|z|^2 + \frac{\tilde{\theta}^2}{4\kappa_0}$$

where

$$c_0 = \min\{c_1, c_2\}, \quad \frac{1}{\kappa_0} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2}.$$

$\Rightarrow x_1, x_2$ bounded since $\dot{V} < 0$ if $|z|^2 > \frac{|\tilde{\theta}|^2}{4\kappa_0 c_0} = \frac{|\tilde{\theta}(0)|^2}{4\kappa_0 c_0}$.

Transient performance improvement with trajectory initialization

$\gamma \neq 0$ also gives

$$\frac{d}{dt}\left(\frac{1}{2}|z|^2\right) \leq -c_0|z|^2 + \frac{\tilde{\theta}^2}{4\kappa_0}.$$

$$\frac{1}{2\gamma}|\tilde{\theta}(t)|^2 \leq \frac{1}{2}|z(t)|^2 + \frac{1}{2\gamma}\tilde{\theta}(t)^2 = V_2(t) \leq V_2(0) = \frac{1}{2}|z(0)|^2 + \frac{1}{2\gamma}\tilde{\theta}(0)^2$$

$$\Downarrow$$

$$||\tilde{\theta}||_\infty^2 \leq \gamma|z(0)|^2 + \tilde{\theta}(0)^2$$

$$\Downarrow$$

$$\frac{d}{dt}\left(|z|^2\right) \leq -2c_0|z|^2 + \frac{1}{2\kappa_0}[\gamma|z(0)|^2 + \tilde{\theta}(0)^2]$$

$$\Downarrow$$

$$|z(t)|^2 \leq |z(0)|^2 e^{-2c_0 t} + \frac{1}{4\kappa_0 c_0}[\gamma|z(0)|^2 + \tilde{\theta}(0)^2].$$

$\Rightarrow$ Transient behaviour depends on c_0, κ_0, γ.
Question: RHS $\searrow$ as $\kappa_0 c_0 \nearrow$?
Answer: No, because it may happen that $|z(0)| \nearrow$ as $\kappa_0 c_0 \nearrow$.
Recall the error variables:

$$z_1 = x_1 - y_r$$

$$z_2 = x_2 - \alpha_1 = x_2 + c_1 z_1 + \kappa_1 z_1 \psi^2 + \hat{\theta}\psi - \dot{y}_r$$

$\Rightarrow$ If $z_1 \neq 0$, it may happen that $|z_2(0)| \nearrow$ as $\kappa_1, c_1 \nearrow$.

Nevertheless, $\exists$ a systematic way to improve transient performance through trajectory initialization (render $z(0) = 0$ independently of parameters by adjusting the initial part of the desired trajectory).

- Set

$$y_r(0) = x_1(0)$$

$$\Downarrow$$

$$z_1(0) = 0$$

$$\Downarrow$$

$$\alpha_1(0) = \dot{y}_r(0) - \hat{\theta}(0)\psi(0).$$

Set

$$\dot{y}_r(0) = x_2(0) + \hat{\theta}(0)\psi(0)$$
$$\Downarrow$$
$$z_2(0) = 0$$
$$\Downarrow$$
$$z(0) = 0$$
$$\Downarrow$$
$$|z(t)|^2 \le \frac{1}{4\kappa_0 c_0}\tilde{\theta}(0)^2$$
$$\Downarrow$$
$$||z(t)||_\infty \le \frac{1}{2\sqrt{\kappa_0 c_0}}|\tilde{\theta}(0)|.$$

$\Rightarrow$ RHS $\searrow$ as $\kappa_0 c_0 \nearrow$.

Chapter 7

Observer Backstepping and Output Feedback

7.1 Observer Backstepping

7.1.1 Unmeasured States

State Feedback Revisited

$$\dot{x} = -x + x^4 + x^2\xi$$
$$\dot{\xi} = -k\xi + u$$

where $k > 0$.

Suppose x, ξ are measurable $\Rightarrow$ backstepping controller

$$\alpha_1(x) = -x^2$$

$$\Downarrow$$

$$\dot{x} = -x + x^2 z, \quad z = \xi - \alpha_1$$
$$\dot{z} = -k\xi + u + 2x(-x + x^4 + x^2\xi) = -k\xi + u + 2x(-x + x^2 z).$$

For $V(x, \xi) = \frac{1}{2}(x^2 + z^2)$,

$$\dot{V} = -x^2 + z\left[x^3 - k\xi + u + 2x(-x + x^2 z)\right].$$

Pick

$$u = -cz - x^3 + k\xi - 2x(-x + x^2 z)$$

$$\Downarrow$$

$$\dot{V} = -x^2 - cz^2$$

$$\Downarrow$$

$x = 0$, $\xi = 0$ is globally asymptotically stable equil. pt of the closed loop system:

$$\dot{x} = -x + x^2 z$$

$$\dot{z} = -cz - x^3.$$

<u>State Estimates as Virtual Controls</u>

Suppose ξ is not measurable $\Rightarrow$ Try Separation Principle

Consider the observer:

$$\dot{\hat{\xi}} = -k\hat{\xi} + u.$$

Define $\tilde{\xi} = \xi - \hat{\xi} \Rightarrow$

$$\dot{\tilde{\xi}} = -k\tilde{\xi} \quad \Rightarrow \quad \tilde{\xi}(t) = \tilde{\xi}(0)e^{-kt}$$

$$\Downarrow$$

$$\dot{x} = -x + x^4 + x^2\hat{\xi} + x^2\tilde{\xi}$$

$$\dot{\hat{\xi}} = -k\hat{\xi} + u$$

$$\dot{\tilde{\xi}} = -k\tilde{\xi}.$$

$\hat{\xi}$ is only possible virtual control $\Rightarrow$

$$z = \hat{\xi} - \alpha_1(x) = \hat{\xi} + x^2$$

$$\Downarrow$$

$$\dot{x} = -x + x^2 z + x^2\tilde{\xi}$$

$$\dot{z} = -cz - x^3 + 2x^3\tilde{\xi}$$

$$\dot{\tilde{\xi}} = -k\tilde{\xi}.$$

Similar to certainty equivalence case, it can be shown that this system possess the finite escape time properties.

Nonlinear Damping

This problem can be handled adding a nonlinear damping term.

Step 1:

$$\alpha_1(x) = -x^2 - s(x)x$$
$$\Downarrow$$
$$\dot{x} = -x + x^2 z - x^3 s(x) + x^2 \xi.$$

Design $s(x)$ using $V(x) = \frac{1}{2}x^2$.

$$\dot{V} = -x^2 + x^3 z - x^2 s(x) + x(x^2)\xi.$$

Pick

$$s(x) = d_1 x^2, \quad d_1 > 0$$
$$\Downarrow$$
$$\alpha_1(x) = -x^2 - d_1 x^3$$
$$z = \xi - \alpha_1(x) = \xi + x^2 + d_1 x^3$$
$$\dot{x} = -x - d_1 x^5 + x^2 z + x^2 \xi.$$
$$\Downarrow$$
$$\dot{V} = -x^2 - d_1 x^6 + x^3 z + x^3 \xi = -x^2 + x^3 z - d_1 \left(x^3 - \frac{1}{2d_1}\xi\right)^2 + \frac{1}{4d_1}\xi^2$$
$$\leq -x^2 + x^3 z + \frac{1}{4d_1}\xi^2.$$

Indeed, better RHS is attained if we add ξ^2 term:

$$V_1(x,\xi) = V(x) + \frac{1}{2d_1 k}\xi^2 = \frac{1}{2}x^2 + \frac{1}{2d_1 k}\xi^2$$
$$\Downarrow$$
$$\dot{V}_1 = \dot{V} - \frac{1}{d_1 k}\xi\dot{\xi} = \dot{V} - \frac{1}{d_1}\xi^2 \leq -x^2 + x^3 z + \frac{1}{4d_1}\xi^2 - \frac{1}{d_1}\xi^2 = -x^2 + x^3 z - \frac{3}{4d_1}\xi^2.$$

Step 2:

$$\dot{z} = -k\xi + u - \frac{d\alpha_1}{dx}(x)(-x + x^4 + x^2\xi)$$

$$= -k\xi + u - \frac{d\alpha_1}{dx}(x)(-x + x^4 + x^2\xi) - \frac{d\alpha_1}{dx}(x)x^2\tilde{\xi}.$$

Again to attain better bound for the Lyapunov function, add to V_1 not only z^2 term but also $\tilde{\xi}^2$ term:

$$V_2(x, z, \tilde{\xi}) = V_1(x, \tilde{\xi}) + \frac{1}{2}z^2 + \frac{1}{2d_2k}\tilde{\xi}^2 = \frac{1}{2}\left[x^2 + z^2 + \left(\frac{1}{d_1k} + \frac{1}{d_2k}\right)\tilde{\xi}^2\right].$$

$$\Downarrow$$

$$\dot{V}_2 \leq -x^2 + x^3z - \frac{3}{4d_1}\tilde{\xi}^2 - \frac{1}{d_2}\tilde{\xi}^2$$

$$+z\left[-k\xi + u - \frac{d\alpha_1}{dx}(x)(-x + x^4 + x^2\xi) - \frac{d\alpha_1}{dx}(x)x^2\tilde{\xi}\right]$$

$$= -x^2 - \frac{3}{4d_1}\tilde{\xi}^2 - \frac{1}{d_2}\tilde{\xi}^2$$

$$+z\left[x^3 - k\xi + u - \frac{d\alpha_1}{dx}(x)(-x + x^4 + x^2\xi)\right] - z\frac{d\alpha_1}{dx}(x)x^2\tilde{\xi}.$$

Pick

$$u = -cz - \underbrace{d_2z\left(\frac{d\alpha_1}{dx}(x)x^2\right)^2}_{\text{nonlinear damping}} - x^3 + k\xi + \frac{d\alpha_1}{dx}(x)(-x + x^4 + x^2\xi)$$

$$\Downarrow$$

$$\dot{V}_2 \leq -x^2 - cz^2 - \frac{3}{4d_1}\tilde{\xi}^2 - d_2z^2\left(\frac{d\alpha_1}{dx}x^2\right)^2 - z\frac{d\alpha_1}{dx}x^2\tilde{\xi} - \frac{1}{d_2}\tilde{\xi}^2$$

$$= -x^2 - cz^2 - \frac{3}{4d_1}\tilde{\xi}^2 - d_2\left(z\frac{d\alpha_1}{dx}x^2 - \frac{1}{2d_2}\tilde{\xi}\right)^2 - \frac{3}{4d_2}\tilde{\xi}^2$$

$$\leq -x^2 - cz^2 - \frac{3}{4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\tilde{\xi}^2.$$

$\Rightarrow$ $x = 0$, $z = 0$, $\tilde{\xi} = 0$ is the globally asymptotically stable equil. pt of the closed loop system:

$$\dot{x} = -x - d_1x^5 + x^2z + x^2\tilde{\xi}$$

$$\dot{z} = -cz - x^3 - d_2z\left(\frac{d\alpha_1}{dx}x^2\right)^2 - \frac{d\alpha_1}{dx}x^2\tilde{\xi}$$

$$\dot{\tilde{\xi}} = -k\tilde{\xi}.$$

7.1.2 Observer Backstepping for Output Feedback Systems

Consider the output feedback form:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \psi_1(y) \\
\dot{x}_2 &= x_3 + \psi_2(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \psi_{\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \psi_\rho(y) + b_m\beta(y)u \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \psi_{n-1}(y) + b_1\beta(y)u \\
\dot{x}_n &= \psi_n(y) + b_0\beta(y)u \\
y &= x_1.
\end{aligned}
$$

Assumptions: $b_m s^m + \cdots + b_1 s + b_0$ is Hurwitz and $\beta(y) \neq 0$ for all y.

Rewrite the system as

$$
\begin{aligned}
\dot{x} &= A_c x + \psi(y) + b\beta(y)u \\
y &= c^T x
\end{aligned}
$$

where

$$
b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \psi(y) = \begin{bmatrix} \psi_1(y) \\ \vdots \\ \psi_n(y) \end{bmatrix}, \quad A_c = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}.
$$

An exponetial observer is

$$
\dot{\hat{x}} = A_c\hat{x} + k(y - \hat{y}) + \psi(y) + b\beta(y)u
$$

$$\hat{y} = c^T \hat{x}$$

where k is chosen so that $A_0 = A_c - kc$ is Hurwitz.

Error variable $\tilde{x} = x - \hat{x} \Rightarrow$

$$\dot{\tilde{x}} = A_0 \tilde{x}.$$

Theorem: Suppose $b_m s^m + \cdots + b_1 s + b_0$ is Hurwitz and $y_r, \dot{y}_r, \cdots, y_r^{(\rho)}$ are known and bounded on $[0, \infty)$ and $y_r^{(\rho)}$ is piecewise continuous. Then $\exists$ feedback control that guarantees global boundedness of $x(t)$ and $\hat{x}(t)$ and regulation of tracking error:

$$\lim_{t\to\infty} [y(t) - y_r(t)] = 0.$$

One choice is

$$u = \frac{1}{b_m \beta(y)} [\alpha_\rho - \hat{x}_{\rho+1} - y_r^{(\rho)}]$$

where

$$z_1 = y - y_r$$

$$z_i = \hat{x}_i - \alpha_{i-1}(y, \hat{x}_1, \cdots, \hat{x}_{i-1}, y_r, \cdots, y_r^{(i-2)}) - y_r^{(i-1)}$$

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \psi_1(y)$$

$$\alpha_i = -c_i z_i - z_{i-1} - d_i \left(\frac{d\alpha_{i-1}}{dy}\right)^2 z_i - k_i(y - \hat{x}_1) - \psi_i(y)$$

$$+\frac{d\alpha_{i-1}}{dy}[\hat{x}_2 + \psi_1(y)] + + \sum_{j=1}^{i-1} \frac{d\alpha_{i-1}}{d\hat{x}_j}[\hat{x}_{j+1} + k_j(y - \hat{x}_1) + \psi_j(y)] + \sum_{j=1}^{i-2} \frac{d\alpha_{i-1}}{dy_r^{(j)}} y_r^{(j+1)}.$$

7.2 Adaptive Observer Backstepping for Parametric Output Feedback Systems

Example:

$$\dot{x}_1 = x_2 + \theta\psi_1(y)$$

$$\dot{x}_2 = x_3 + \theta\psi_2(y) + u$$

$$\dot{x}_3 = u$$

$$y = x_1.$$

x_2, x_3: unmeasurable

Goal: Design an adaptive nonlinear output feedback controller that guarantees asymptotic tracking and boundedness of all states.

Rewrite the plant:

$$\dot{x} = \boldsymbol{A}x + bu + \theta\psi(y),$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \psi(y) = \begin{bmatrix} \psi_1(y) \\ \psi_2(y) \\ 0 \end{bmatrix}.$$

If θ is known, the observer would be

$$\dot{\hat{x}} = \boldsymbol{A}\hat{x} + bu + \theta\psi(y) + k(y - \hat{y}) = \boldsymbol{A}\hat{x} + bu + \theta\psi(y) + k(y - c\hat{x}) = \boldsymbol{A}_0\hat{x} + bu + \theta\psi(y) + ky$$

where

$$\boldsymbol{A}_0 = \boldsymbol{A} - kc = \begin{bmatrix} -k_1 & 1 & 0 \\ -k_2 & 0 & 1 \\ -k_3 & 0 & 0 \end{bmatrix} \quad \text{is Hurwitz.}$$

However, θ is unknown

$$\dot{\hat{x}} = \boldsymbol{A}_0\hat{x} + bu + ky + \theta\psi(y).$$

To reconstruct $\hat{x}$, break it up in two parts:

$$\hat{x} = \xi_0 + \theta\xi_1$$

$$\Downarrow$$

$$\dot{\xi}_0 = \boldsymbol{A}_0\xi_0 + ky + bu, \quad \dot{\xi}_1 = \boldsymbol{A}_0\xi_1 + \psi(y).$$

Error variable: $\epsilon = x - \hat{x} = x - \xi_0 - \theta\xi_1 \Rightarrow$

$$\dot{\epsilon} = \dot{x} - \dot{\xi}_0 - \theta\dot{\xi}_1 = \boldsymbol{A}_0 x + ky + bu + \theta\psi(y) - \boldsymbol{A}_0\xi_0 - ky - bu - \boldsymbol{A}_0\theta\xi_1 - \theta\psi(y)$$

$$= \boldsymbol{A}_0(x - \xi_0 - \theta\xi_1) = \boldsymbol{A}_0\epsilon.$$

$\boldsymbol{A}_0$ Hurwitz $\Rightarrow$

$$x = \xi_0 + \theta\xi_1 + \epsilon$$

where ϵ converges to zero exponentially.

Step 1: Tracking error: $z_1 = y - y_r$

$$\dot{z}_1 = \dot{y} - \dot{y}_r = x_2 + \theta\psi_1(y) - \dot{y}_r = \xi_{02} + \theta\xi_{12} + \epsilon_2 + \theta\psi_1(y) - \dot{y}_r = \xi_{02} + \theta[\psi_1(y) + \xi_{12}] + \epsilon_2.$$

The only candidate for virtual control is ξ_{02}.

The second error variable:

$$z_2 = \xi_{02} - \alpha_1 - \dot{y}_r$$

$$\Downarrow$$

$$\dot{z}_1 = z_2 + \alpha_1 + \theta w + \epsilon_2.$$

Pick

$$\alpha_1 = -c_1 z_1 + z_2 - d_1 z_1 - \vartheta_1 w$$

where ϑ_1 is the first estimate of θ.

$$\Downarrow$$

$$\dot{z}_1 = -c_1 z_1 + z_2 - d_1 z_1 + (\theta - \vartheta_1) w + \epsilon_2.$$

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\gamma}(\theta - \vartheta_1)^2 + \frac{1}{d_1}\epsilon^T \boldsymbol{P}_0 \epsilon$$

where $\gamma > 0$, $\boldsymbol{P}_0 = \boldsymbol{P}_0^T > 0$ and $\boldsymbol{P}_0 \boldsymbol{A}_0 + \boldsymbol{A}_0^T \boldsymbol{P}_0 = -\boldsymbol{I}$.

$$\Downarrow$$

$$\begin{aligned}
\dot{V}_1 &= z_1 \dot{z}_1 - \frac{1}{\gamma}(\theta - \vartheta_1)\dot{\vartheta}_1 + \frac{1}{d_1}\frac{1}{dt}\left(\epsilon^T \boldsymbol{P}_0 \epsilon\right) \\
&= z_1 z_2 - c_1 z_1^2 - d_1 z_1^2 + (\theta - \vartheta_1)\left(z_1 w - \frac{1}{\gamma}\vartheta_1\right) + z_1 \epsilon_2 - \frac{1}{d_1}\epsilon^T \epsilon \\
&= z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1)\left(z_1 w - \frac{1}{\gamma}\vartheta_1\right) \\
&\quad - d_1 \left[z_1 - \frac{1}{2d_1}\epsilon_2\right]^2 + \frac{1}{4d_1}\epsilon_2^2 - \frac{1}{d_1}\epsilon^T \epsilon \\
&\le z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1)\left(z_1 w - \frac{1}{\gamma}\vartheta_1\right) - \frac{3}{4d_1}\epsilon^T \epsilon.
\end{aligned}$$

Choose

$$\vartheta_1 = \gamma z_1 w.$$

Step 2:

$$\begin{aligned}
\dot{z}_2 &= \dot{\xi}_{02} - \dot{\alpha}_1 - \ddot{y}_r \\
&= u + k_2(y - \xi_{01}) + \xi_{03} - \frac{d\alpha_1}{dy}\dot{y} \\
&\quad -\frac{d\alpha_1}{d\xi_{12}}(-k_2\xi_{11} + \xi_{13} + \psi_2(y)) - \frac{d\alpha_1}{dy_r}\dot{y}_r - \frac{d\alpha_1}{d\vartheta_1}\dot{\vartheta}_1 - \ddot{y}_r \\
&= u + k_2(y - \xi_{01}) + \xi_{03} - \frac{d\alpha_1}{dy}(\xi_{02} + \theta w + \epsilon_2) \\
&\quad -\frac{d\alpha_1}{d\xi_{12}}(-k_2\xi_{11} + \xi_{13} + \psi_2(y)) - \frac{d\alpha_1}{dy_r}\dot{y}_r - \frac{d\alpha_1}{d\vartheta_1}\gamma w z_1 - \ddot{y}_r \\
&= u + k_2(y - \xi_{01}) + \xi_{03} - \frac{d\alpha_1}{dy}\xi_{02} - \frac{d\alpha_1}{d\xi_{12}}(-k_2\xi_{11} + \xi_{13} + \psi_2(y)) \\
&\quad -\frac{d\alpha_1}{dy_r}\dot{y}_r - \frac{d\alpha_1}{d\vartheta_1}\gamma w z_1 - \frac{d\alpha_1}{dy}\theta w - \frac{d\alpha_1}{dy}\epsilon_2 - \ddot{y}_r.
\end{aligned}$$

Pick

$$\begin{aligned}
u &= -c_2 z_2 - z_1 - d_2\left(\frac{d\alpha_1}{dy}\right)^2 z_2 - k_2(y - \xi_{01}) - \xi_{03} + \frac{d\alpha_1}{dy}(\xi_{02} + \vartheta_2 w) \\
&\quad +\frac{d\alpha_1}{d\xi_{12}}(-k_2\xi_{11} + \xi_{13} + \psi_2(y)) + \frac{d\alpha_1}{dy_r}\dot{y}_r + \frac{d\alpha_1}{d\vartheta_1}\gamma w z_1 + \ddot{y}_r
\end{aligned}$$

where ϑ_2 is new estimate of θ and the third term on RHS is nonlinear damping for the disturbance ϵ_2.

$$\Downarrow$$

$$\dot{z}_2 = -c_2 z_2 - z_1 - \frac{d\alpha_1}{dy}w(\theta - \vartheta_2) - d_2\left(\frac{d\alpha_1}{dy}\right)^2 z_2 - \frac{d\alpha_1}{dy}\epsilon_2.$$

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(\theta - \vartheta_2)^2 + \frac{1}{d_2}\epsilon^T P_0 \epsilon$$

$$\Downarrow$$

$$\dot{V}_2 = \dot{V}_1 + z_2\dot{z}_2 - \frac{1}{\gamma}(\theta - \vartheta_2)\dot{\vartheta}_2 - \frac{1}{d_2}\epsilon^T\epsilon$$

$$\leq z_1 z_2 - c_1 z_1^2 - \frac{3}{4d_1}\epsilon^T\epsilon - c_2 z_2^2 - z_1 z_2$$

$$-\left(\frac{d\alpha_1}{dy} w z_2 + \frac{1}{\gamma}\dot{\vartheta}_2\right)(\theta - \vartheta_2) - d_2\left(\frac{d\alpha_1}{dy}\right)^2 z_2^2 - \frac{d\alpha_1}{dy} z_2 \epsilon_2 - \frac{1}{d_2}\epsilon^T\epsilon$$

$$\leq -c_1 z_1^2 - c_2 z_2^2 - \left(\frac{3}{4d_1} + \frac{3}{4d_2}\right)\epsilon^T\epsilon - \left(\frac{d\alpha_1}{dy} w z_2 + \frac{1}{\gamma}\dot{\vartheta}_2\right)(\theta - \vartheta_2).$$

Pick

$$\dot{\vartheta}_2 = -\gamma \frac{d\alpha_1}{dy} w z_2$$

$$\Downarrow$$

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 - \frac{3}{4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\epsilon^T\epsilon.$$

$\Rightarrow$ $z_1, z_2, \vartheta_1, \vartheta_2, \epsilon$ bounded and $z_1, z_2, \epsilon \to 0$ as $t \to \infty$.

$\Rightarrow$ $y = x_1 = z_1 + y_r$ bounded $\Rightarrow$ ξ_1 bounded because $\dot{\xi}_1 = A_0\xi_1 + \psi(y)$

$\Rightarrow$ $\xi_{01} = x_1 - \epsilon_1 - \theta\xi_{11}$ bounded.

$w = \psi_1(y) + \xi_{12}$ bounded $\Rightarrow$ $\alpha_1 = -c_1 z_1 - d_1 z_1 - \vartheta_1 w$ bounded

$\Rightarrow$ $\xi_{02} = z_2 + \alpha_1 + \dot{y}_r$ and $x_2 = \epsilon_2 + \xi_{02} + \theta\xi_{12}$ bounded.

Define $\zeta = x_3 - x_2 + y$.

$$\dot{\zeta} = \dot{x}_3 - \dot{x}_2 + \dot{x}_1 = u - x_3 - \theta\psi_2(y) - u + x_2 + \theta\psi_1(y)$$

$$= -\zeta + y\theta[\psi_1(y) - \psi_2(y)].$$

y bounded $\Rightarrow$ ζ bounded $\Rightarrow$ $x_3 = \zeta + x_2 - y$ and $\xi_{03} = x_3 - \epsilon_3 - \theta\xi_{13}$ bounded

$\Rightarrow$ u bounded.

Consider the parametric output feedback systems:

$$\dot{x}_1 = x_2 + \psi_{01}(y) + \sum_{j=1}^{p}\theta_j\psi_{j,1}(y)$$

$$\dot{x}_2 = x_3 + \psi_{02}(y) + \sum_{j=1}^{p}\theta_j\psi_{j,2}(y)$$

$$\vdots$$

$$\dot{x}_{\rho-1} = x_\rho + \psi_{0,\rho-1}(y) + \sum_{j=1}^{p}\theta_j\psi_{j,\rho-1}(y)$$

$$\dot{x}_\rho = x_{\rho+1} + \psi_{0,\rho}(y) + \sum_{j=1}^{p} \theta_j \psi_{j,\rho}(y) + b_m \beta(y) u$$

$$\vdots$$

$$\dot{x}_n = \psi_{0,n}(y) + \sum_{j=1}^{p} \theta_j \psi_{j,n}(y) + b_0 \beta(y) u$$

$$y = x_1$$

where $\theta_1, \cdots, \theta_p$ and $b_0, \cdots, b_m$ are unknown constants.

Assumptions:

- The sign of b_m is known.
- The polynomial $B(s) = b_m s^m + \cdots + b_1 s + b_0$ is Hurwitz.
- $\beta(y) \neq 0$ for all y
- The reference signal y_r and its first ρ derivatives are konwn and bounded and $y_r^{(\rho)}(t)$ is piecewise continuous.

Rewrite the plant as:

$$\dot{x} = A_c x + \psi_0(y) + \sum_{j=1}^{p} \theta_j \psi_j(y) + b\beta(y) u$$

$$y = c^T x$$

where

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\psi_j(y) = [\psi_{j,1}(y), \cdots, \psi_{j,n}(y)]^T.$$

Choose k such that $X_0 = A - kc^T$ is Hurwitz and define

$$\dot{\xi}_0 = A_0 \xi_0 + ky + \psi_0(y)$$

$$\dot{\xi}_j = A_0 \xi_j + \psi_j(y)$$
$$\dot{v}_j = A_0 v_j + e_{n-j}\beta(y)u$$

where e_i is ith coordinate vector.

Then

$$\hat{x} = \xi_0 + \sum_{j=1}^{p} \theta_j \xi_j + \sum_{j=0}^{m} b_j v_j.$$

Note that

$$v_j = (A_0)^j \lambda$$

where

$$\dot{\lambda} = A_0 \lambda + e_n \beta(y) u.$$

Define observer error variable:

$$\epsilon := x - \left(\xi_0 + \sum_{j=1}^{p} \theta_j \xi_j + \sum_{j=0}^{m} b_j v_j \right)$$
$$\Downarrow$$
$$\dot{\epsilon} = A_0 \epsilon.$$

Theorem: Under the assumptions,

$$u = \frac{1}{\beta(y)}[\alpha_\rho - v_{m,\rho+1} + \vartheta_r^{(\rho)}$$

$$\dot{\vartheta}_1 = sgn(b_m)\Gamma[w_1(y, \bar{\xi}^{(2)}, \bar{v}^{(2)}, \bar{y}_r^{(1)}) - \dot{y}_r e_1] z_1$$
$$\dot{\vartheta}_2 = \Gamma[w_2(y, \bar{\xi}^{(2)}, \bar{v}^{(2)}, \bar{\vartheta}^{(2)}, \bar{y}_r^{(1)}) - z_1 e_{p+m+1}] z_2$$
$$\dot{\vartheta}_i = \Gamma w_i(y, \bar{\xi}^{(i)}, \bar{v}^{(i)}, \bar{\vartheta}^{(i)}, \bar{y}_r^{(i-1)}) z_i$$

where

$$z_1 = y - y_r$$
$$z_i = v_{m,i} - \alpha_{i-1}(y, \bar{\xi}^{(i)}, \bar{v}^{(i)}, \bar{\vartheta}^{(i)}, \bar{y}_r^{(i-1)}) - \vartheta_{1,1} y_r^{(i)}$$
$$\alpha_1 = -\vartheta_1^T w_1$$
$$\alpha_2 = -c_2 z_2 - \vartheta_{2,p+m+1} z_1 - d_2 \left(\frac{d\alpha_1}{dy}\right)^2 z_2 + \frac{d\alpha_1}{dy}[\xi_{0,2} + \psi_{0,1}(y)] - \vartheta_2^T w_2$$

$$+k_2 v_{m,1} + \frac{d\alpha_1}{d\xi_0}[A_0\xi_0 + ky + \psi_0(y)] + \sum_{j=1}^{p}\frac{d\alpha_1}{d\xi_j}[A_0\xi_j + \psi_j(y)]$$

$$+\sum_{j=0}^{m}\frac{d\alpha_1}{dv_j}A_0 v_j + \left(\frac{d\alpha_1}{d\vartheta_1} + \dot{y}_r e_1^T\right) sgn(b_m)\Gamma[w_1 - \dot{y}_r e_1]z_1 + \frac{d\alpha_1}{dy_r}\dot{y}_r$$

$$\alpha_i = -c_i z_i - z_{i-1} - d_i\left(\frac{d\alpha_{i-1}}{dy}\right)^2 z_i + \frac{d\alpha_{i-1}}{dy}[\xi_{0,2} + \psi_{0,1}(y)] - \vartheta_i^T w_i$$

$$+k_i v_{m,1} + \frac{d\alpha_{i-1}}{d\xi_0}[A_0\xi_0 + ky + \psi_0(y)] + \sum_{j=1}^{p}\frac{d\alpha_{i-1}}{d\xi_j}[A_0\xi_j + \psi_j(y)]$$

$$+\sum_{j=0}^{m}\frac{d\alpha_{i-1}}{dv_j}A_0 v_j + \left(\frac{d\alpha_{i-1}}{d\vartheta_1} + y_r^{(i-1)} e_1^T\right) sgn(b_m)\Gamma[w_1 - \dot{y}_r e_1]z_1$$

$$+\frac{d\alpha_{i-1}}{d\vartheta_2}\Gamma[w_2 + z_1 e_{p+m+1}]z_2 + \sum_{j=3}^{i-1}\frac{d\alpha_{i-1}}{d\vartheta_j}\Gamma w_j z_j + \sum_{j=1}^{i-2}\frac{d\alpha_{i-1}}{dy_r^{(j)}}y_r^{(j+1)}$$

$$w_1^T = [c_1 z_1 + d_1 z_1 + \xi_{0,2} + \psi_{0,1}, \psi_{1,1} + \xi_{1,2}, \cdots, \psi_{p,1} + \xi_{p,2}, v_{0,2}, \cdots, v_{m-1,2}]$$

$$w_i^T = -\frac{d\alpha_{i-1}}{dy}[\psi_{1,1} + \xi_{1,2}, \cdots, \psi_{p,1} + \xi_{p,2}, v_{0,2}, \cdots, v_{m-1,2}, v_{m,2}]$$

$$\bar{\xi}^{(i)} = [\xi_{0,1}, \cdots, \xi_{0,i}, \cdots, \xi_{p,1}, \cdots, \xi_{p,i}]$$

$$\bar{v}^{(i)} = [v_{0,1}, \cdots, v_{0,i}, \cdots, v_{m-1,1}, \cdots, v_{m-1,i}, v_{m,1}, \cdots, v_{m,i-1}]$$

$$\bar{\vartheta}^{(i)} = [\vartheta_1^T, \cdots, \vartheta_i^T]$$

$$\bar{y}_r^{(i)} = [y_r, \dot{y}_r, \cdots, y_r^{(i)}]$$

guarantees global boundedness of $x(t), \xi_0(t), \cdots, \xi_p(t)$ and $v_0(t), \cdots, v_m(t)$ and

$$\lim_{t\to\infty}[y(t) - y_r(t)] = 0.$$