Process Control

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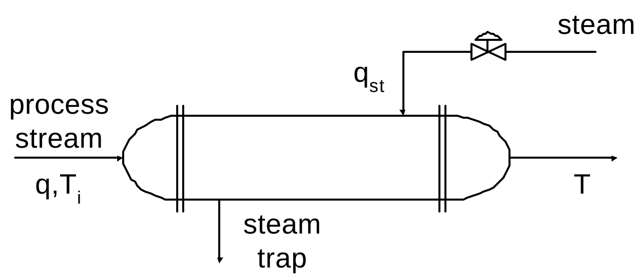
Chapter 1

Introduction

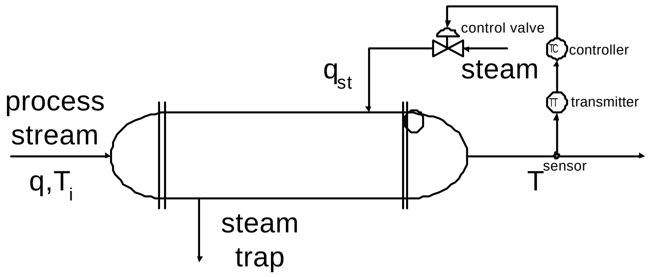
The chemical processes are designed to produce the product with desired specifications. However, due to the effect of exogeneous disturbances, the properties of the product deviate from the desired specifications during the operation. This deviation can be partly reduced by adjusting some input to the system. The goal of automatic control is to design a system called controller that detects the deviation and automatically manipulates the control input to reduce the deviation created by the exogeneous disturbances.

Ex: Process (Heat Exchanger)

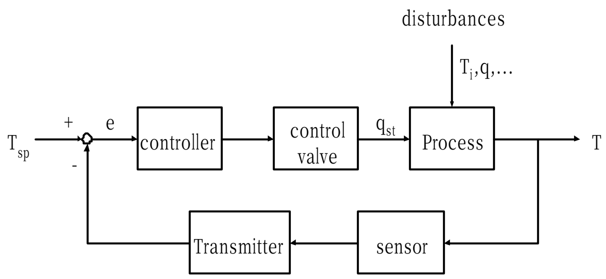


Objective: Maintain the outlet temperature of the process at its desired value in spite of disturbances such as variation of flow rate or temperature of the process stream.

Closed Loop System:



Terminologies



- q_{sf}: manipulated (input, control) variable
- T: controlled (output) variable
- T_i, q: disturbances
- T_{sp}: set point
- e: error

Chapter 2

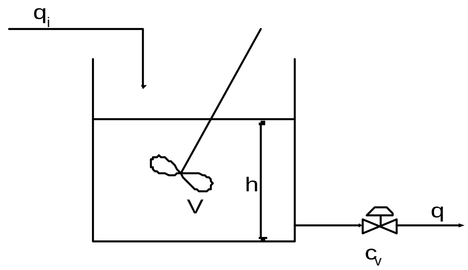
Modeling of Chemical Process Dynamics

The dynamic equation for a process is obtained through balance equation:

$$\left\{\begin{matrix}\text{rate of}\\ \text{accumulation}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of}\\ \text{in}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of}\\ \text{out}\end{matrix}\right\} + \left\{\begin{matrix}\text{rate of}\\ \text{generation}\end{matrix}\right\}$$

2.1 Models of Typical Chemical Processes

Ex1: Liquid Storage Tank



ρ: density of fluid
V: volume of fluid
q_i, q: inlet and outlet volumetric flow rates
A: cross sectional area of tank
h: height of fluid
P: pressure at the bottom of tank
P_a: ambient pressure
C_v: valve coeffient
Assumptions:
i) ρ constant
ii) Outlet flow is discharged at the ambient pressure
Mass balance:

$$\left\{\begin{matrix}\text{rate of mass}\\ \text{accumulation}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of mass}\\ \text{in}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of mass}\\ \text{out}\end{matrix}\right\}$$

$$\frac{d(V\rho)}{dt} = q_i\rho - q\rho.$$

Since $V = Ah$,

$$A\frac{dh}{dt} = q_i - q.$$

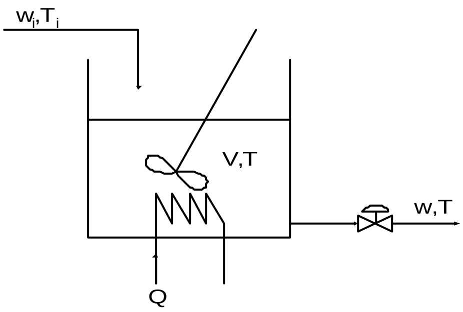
The pressure drop across the valve is proportional to q^2. Hence,

$$q = C_v\sqrt{P - P_a} = C_v\sqrt{\frac{\rho g}{g_c}h}.$$

$$\Downarrow$$

$$A\frac{dh}{dt} = q_i - C_v\sqrt{\frac{\rho g}{g_c}h}.$$

Ex2: Heater



ρ: density of fluid
C: heat capacity of fluid
V: volume of fluid
T_i, T: temperatures of inlet fluid and fluid inside the reactor
T_{ref}: reference temperature
w_i, w: inlet and outlet mass flow rates
Q: heat input by electrical heater
Assumptions: ρ, C constant, perfectly mixed, perfectly insulated
Similar to the liquid storage tank case, the mass balance boils down to

$$\rho \frac{dV}{dt} = w_i - w$$

Energy balance:

$$\left\{\begin{matrix}\text{rate of energy}\\ \text{accumulation}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of energy in}\\ \text{by flow or convection}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of energy out}\\ \text{by flow or convection}\end{matrix}\right\}$$
$$+ \left\{\begin{matrix}\text{net rate of heat addition to}\\ \text{system from surroundings}\end{matrix}\right\}$$
$$C\frac{d[V\rho(T - T_{ref})]}{dt} = w_i C(T_i - T_{ref}) - wC(T - T_{ref}) + Q.$$

However,

$$C\frac{d[V\rho(T-T_{ref})]}{dt} = \rho C(T-T_{ref})\frac{dV}{dt} + V\rho C\frac{d(T-T_{ref})}{dt}$$

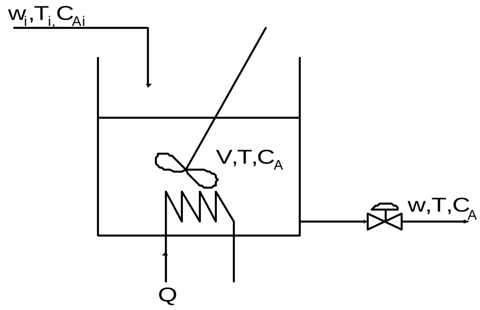
$$= C(T-T_{ref})(w_i - w) + V\rho C\frac{dT}{dt}.$$

To this end,

$$\frac{dV}{dt} = \frac{1}{\rho}(w_i - w),$$

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q}{V\rho C}.$$

Ex3: Continuous Stirred Tank Reactor (CSTR)



ρ: density of fluid
C: heat capacity of fluid
V: volume of fluid
T_i, T: temperatures of inlet fluid and fluid inside the reactor
C_{Ai}, C_A: concentrations of A in inlet stream and inside the reactor
T_{ref}: reference temperature
w_i, w: inlet and outlet mass flow rates
Q: heat input by electrical heater
$(\triangle H)_{rxn}$: heat of reaction
Reaction: $A \to B$

Assumptions: standard assumptions for CSTR + A and B have the same ρ, C, ρ, C constant, perfect insulation, the reaction kinetics is

$$\left(\frac{dC_A}{dt}\right)_{rsn} = -kC_A$$

where

$$k = k_0 e^{-\frac{E}{RT}}.$$

Similar to the heater case, the total mass boils down to

$$\rho\frac{dV}{dt} = w_i - w.$$

In the energy balance, we have the term of heat generated by reaction:

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q + Q_{rsn}}{V\rho C}.$$

where

$$Q_{rsn} = (\triangle H)_{rsn}\left(\frac{dC_A}{dt}\right)_{rsn} = -(\triangle H)_{rsn} k_0 e^{-\frac{E}{RT}} C_A.$$

Component balance:

$$\begin{Bmatrix}\text{rate of component } A\\ \text{accumulation}\end{Bmatrix} = \begin{Bmatrix}\text{rate of component } A \text{ in}\\ \text{by flow or convection}\end{Bmatrix}$$
$$- \begin{Bmatrix}\text{rate of component } A \text{ out}\\ \text{by flow or convection}\end{Bmatrix} + \begin{Bmatrix}\text{rate of generation of } A\\ \text{by reaction}\end{Bmatrix}$$

$$\frac{d[VC_A]}{dt} = \frac{w_i}{\rho}C_{Ai} - \frac{w}{\rho}C_A + V\left(\frac{dC_A}{dt}\right)_{rsn}.$$

However,

$$\frac{d[VC_A]}{dt} = V\frac{dC_A}{dt} + C_A\frac{dV}{dt} = V\frac{dC_A}{dt} + C_A\frac{(w_i - w)}{\rho}.$$

To this end,

$$\rho\frac{dV}{dt} = w_i - w.$$

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q}{V\rho C} - \frac{(\triangle H)_{rsn}}{V\rho C} k_0 e^{-\frac{E}{RT}} C_A.$$

$$\frac{dC_A}{dt} = \frac{w_i}{\rho V}(C_{Ai} - C_A) + k_0 e^{-\frac{E}{RT}} C_A.$$

2.2 Linearization of Nonlinear Systems

Consider the process described by the nonlinear ordinary differential equation:

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

where y is the output, x the state and u input.

Suppose u_{ss} represent the steady state input to the process. Then the steady state x_{ss} is attained when

$$0 = f(x_{ss}, u_{ss}).$$

The steady state output y_{ss} is $h(x_{ss})$.

Approximate f, h around steady state using Taylor series expansion:

$$\begin{aligned}
f(x,u) = \underbrace{f(x_{ss},u_{ss})}_{=0} + \left[\frac{\partial f}{\partial x}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}) + \left[\frac{\partial f}{\partial u}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (u - u_{ss}) \\
+\frac{1}{2}(x - x_{ss})^T \left[\frac{\partial^2 f}{\partial x^2}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}) + (x - x_{ss})^T \left[\frac{\partial^2 f}{\partial x \partial u}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (u - u_{ss}) \\
+\frac{1}{2}(u - u_{ss})^T \left[\frac{\partial^2 f}{\partial u^2}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (u - u_{ss}) + \cdots \\
\approx \left[\frac{\partial f}{\partial x}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}) + \left[\frac{\partial f}{\partial u}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (u - u_{ss}).
\end{aligned}$$

$$\begin{aligned}
h(x) = h(x_{ss},u_{ss}) + \left[\frac{dh}{dx}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}) + \frac{1}{2}(x - x_{ss})^T \left[\frac{d^2 h}{dx^2}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}) + \cdots \\
\approx \underbrace{h(x_{ss},u_{ss})}_{=y_{ss}} + \left[\frac{dh}{dx}\right]_{\substack{x=x_{ss}\\u=u_{ss}}} (x - x_{ss}).
\end{aligned}$$

These linear approximations are usually good valid near the steady state only.

Define deviation variables:

$$y' = y - y_{ss}, \quad x' = x - x_{ss}, \quad y' = y - y_{ss}.$$

Then the approximate linearization of the process is

$$\frac{dx'}{dt} = \boldsymbol{A}x' + bu'$$

$$y' = c^T x'$$

where

$$\boldsymbol{A} = \left[\frac{\partial f}{\partial x}\right]_{\substack{x=x_{ss}\\ u=u_{ss}}}, \quad b = \left[\frac{\partial f}{\partial u}\right]_{\substack{x=x_{ss}\\ u=u_{ss}}}, \quad c^T = \left[\frac{dh}{dx}\right]_{\substack{x=x_{ss}\\ u=u_{ss}}}.$$

Ex: Temperature control of CSTR with $w_i = w$

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q}{V\rho C} - \frac{(\triangle H)_{rxn}}{V\rho C}k_0 e^{-\frac{E}{RT}} C_A =: f_1(T, C_A, Q).$$

$$\frac{dC_A}{dt} = \frac{w_i}{\rho V}(C_{Ai} - C_A) + k_0 e^{-\frac{E}{RT}} C_A =: f_2(T, C_A, Q).$$

Define

$$x_1 = T, \ \ x_2 = C_A, \ \ u = Q, \ \ y = T =: h(x).$$

Then

$$x'_1 = T_{ss}, \ \ x'_2 = C_{Ass}, \ \ u' = Q_{ss}, \ \ y' = T_{ss}.$$

Now

$$\frac{dx'}{dt} = \boldsymbol{A}x' + bu'$$

$$y' = c^T x'$$

where

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial f_1}{\partial T} & \frac{\partial f_1}{\partial C_A} \\ \frac{\partial f_2}{\partial T} & \frac{\partial f_2}{\partial C_A} \end{bmatrix}_{\substack{T\,=\,T_{ss}\\ C_A\,=\,C_{Ass}\\ Q\,=\,Q_{ss}}}$$

$$= \begin{bmatrix} -\frac{w_i}{V\rho} + \frac{(\Delta H)_{rxn}}{V\rho C}\frac{E}{RT_{ss}^2}k_0 e^{-\frac{E}{RT_{ss}}} C_{Ass} & -\frac{(\Delta H)_{rxn}}{V\rho C}k_0 e^{-\frac{E}{RT_{ss}}} \\ -\frac{E}{RT_{ss}^2}k_0 e^{-\frac{E}{RT_{ss}}} C_{Ass} & -\frac{w_i}{\rho V} + k_0 e^{-\frac{E}{RT_{ss}}} \end{bmatrix},$$

$$b = \begin{bmatrix} \frac{\partial f_1}{\partial Q} \\ \frac{\partial f_2}{\partial Q} \end{bmatrix}_{\substack{T\,=\,T_{ss}\\ C_A\,=\,C_{Ass}\\ Q\,=\,Q_{ss}}} = \begin{bmatrix} \frac{1}{V\rho C} \\ 0 \end{bmatrix}, \ \ c^T = \left[\frac{\partial h}{\partial T}\ \frac{\partial h}{\partial T}\right]_{\substack{T=T_{ss}\\ C_A=C_{Ass}}} = [1\ 0].$$

Part I

Signals

Chapter 3

Introduction to Signals

Loosely speaking, signal is a quantitative phenomenon that varies with time. Hence, the signal is taken to be a time function defined on $(-\infty, \infty)$. Often we are only interested in the future from the present. In this case, assuming the present is $t = 0$ without loss of generality, a signal is a function defined on $[0, \infty)$ or is treated without loss of generality as a time function on $(-\infty, \infty)$ which is 0 for all $t < 0$.

Ex: Temperature in the heater

Concentration in the reactor

Position and velocity of robot arm

Chapter 4

Periodic Signals and Fourier Series

A signal u is called periodic with period T if

$$u(t) = u(t \pm kT), \quad \forall k = 0, \pm 1, \pm 2, \cdots .$$

In this case, it suffices to consider any interval with length T; e.g. $[0, T]$, $\left[-\frac{T}{2}, \frac{T}{2}\right]$.

Complex Exponentials

Consider the complex exponential:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

where

$$\omega = \frac{2\pi}{T}.$$

Then

$$e^{j\omega(t+T)} = e^{j\omega t} \underbrace{e^{j2\pi}}_{=1} = e^{j\omega t}.$$

Hence, $e^{j\omega t}$ is periodic with period T. Clearly, ω is the frequency associated with $e^{j\omega t}$.

Now consider the complex exponentials:

$$e^{jn\omega t}, \quad n = 0, \pm 1, \pm 2, \cdots .$$

Then

$$e^{jn\omega(t+T)} = e^{jn\omega t}\underbrace{e^{j2n\pi}}_{=1} = e^{jn\omega t}$$

and thus $e^{jn\omega t}$ is also periodic with period T. Indeed the smallest period of $e^{jn\omega t}$ is $\frac{T}{|n|}$ and the associated frequency is $|n|\omega$.

Define the inner product between two periodic complex functions f, g with period T as

$$\langle f, g\rangle := \int_{-\frac{T}{2}}^{\frac{T}{2}} f\overline{g}dt.$$

f and g are said to be orthogonal if $\langle f, g\rangle = 0$. Now the inner product between two complex exponentials is

$$\langle e^{jm\omega t}, e^{jn\omega t}\rangle := \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{jm\omega t}\overline{e^{jn\omega t}}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(m-n)\omega t}dt$$

$$= \begin{cases} t|_{-\frac{T}{2}}^{\frac{T}{2}} = T & \text{for } m = n \\ \frac{e^{j(m-n)\omega t}}{j(m-n)\omega}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} = T\frac{\sin(m-n)\pi}{(m-n)\pi} = 0 & \text{for } m \neq n \end{cases}.$$

Hence, the above complex exponentials are orthogonal each other.

Fourier Series

Recall that any vector x in $\mathbf{R}^n$ can be expanded with the orthogonal basis vectors $\{v_k\}_{k=1}^n$:

$$x = \sum_{k=1}^{n} a_k v_k$$

From the orthogonality,

$$a_k = \frac{\langle x, v_k\rangle}{\langle v_k, v_k\rangle}.$$

Similarly, a periodic complex function f with period T can be expanded with the basis functions $\left\{e^{jk\omega t}\right\}_{k=-\infty}^{\infty}$:

$$f = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega t}$$

where

$$F_k = \frac{\langle f, e^{jk\omega t}\rangle}{\langle e^{jk\omega t}, e^{jk\omega t}\rangle} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\omega t}dt.$$

The above expansion is called the Fourier series expansion and F_k's are called the Fourier coefficients. Notice that

$$F_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt.$$

Remark 1: Given f, the Fourier coefficients that form a sequence (function of integers) can be computed using the Fourier coefficient formula. Conversely given Fourier coefficients, the original function f can be recovered from them. Hence the Fourier series defines an invertible relationship between a periodic function and the associated Fourier coefficient sequence.

Remark 2: Visible light composes of various light with different wave lengths or frequencies. Using the prism, visible light can be decomposed into light with different frequencies. Similarly, a periodic function composes periodic functions with different frequencies and can be decomposed into periodic functions with different frequencies via Fourier series expansion.

Ex: Let $T = 2$ and

$$f(t) = t, \qquad t \in [-1, 1].$$

Hence $\omega = \pi$. Now the Fourier coefficients are

$$F_0 = \frac{1}{2}\int_{-1}^{1} t dt = 0$$

and for $k = \pm 1, \pm 2, \cdots$,

$$F_k = \frac{1}{2}\int_{-1}^{1} te^{-jk\pi t}dt = \underbrace{\left[-\frac{1}{2jk\pi}te^{-jk\pi t}\right]_{t=-1}^{1}}_{=-\frac{1}{jk\pi}\cos k\pi} + \frac{1}{2jk\pi}\int_{-1}^{1} e^{-jk\pi t}dt$$

$$= -\frac{1}{jk\pi}(-1)^k - \underbrace{\frac{1}{2k^2\pi^2}e^{-jk\pi t}\Big|_{t=-1}^{1}}_{=0} = (-1)^k\frac{j}{k\pi}.$$

To this end,

$$f(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (-1)^k \frac{j}{k\pi}e^{jk\pi t}.$$

Spectra of Signals

The Fourier coefficient of a frequency represents how much the associated frequency component, $e^{jk\omega t}$, the original periodic function has. However, the Fourier coefficients are complex scalars. Hence in polar form,

$$F_k = |F_k|e^{j\theta_k}$$

where

$$|F_k| = \sqrt{[Re(F_k)]^2 + [Im(F_k)]^2}$$

$$\tan\theta_k = \frac{Im(F_k)}{Re(F_k)}.$$

$|F_k|$ as a function of k is called the amplitude spectrum whereas θ_k as a function of k the phase spectrum. Notice that the spectra can be viewed as a function of frequency $k\omega$ instead of k.

The signals we will consider are real. Hence, let f be a real periodic function. Then $f = \overline{f}$ and thus

$$F_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\omega t}dt = \overline{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jk\omega t}dt} = \overline{F_{-k}}.$$

Hence,

$$|F_k|e^{j\theta_k} = F_k = \overline{F_{-k}} = \overline{|F_{-k}|e^{j\theta_{-k}}} = |F_{-k}|e^{-j\theta_{-k}}.$$

To this end for real periodic signals, the amplitude spectrum is an even function of k whereas the phase spectrum is an odd function of k. Hence for real periodic signals, it suffices to examine the spectra for $k \geq 0$.

Parseval's Theorem

Parseval's Theorem: Let f be a periodic signal. Then

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |F_k|^2.$$

Proof:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\overline{f(t)}dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} F_k\overline{F_l}e^{j(k-l)\omega t}dt = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} F_k\overline{F_l}\{\frac{1}{T}\underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j(k-l)\omega t}dt}_{=\begin{cases} T & \text{for } k=l \\ 0 & \text{for } k\neq l\end{cases}}\}$$

$$= \sum_{k=-\infty}^{\infty} F_k \overline{F_k} = \sum_{k=-\infty}^{\infty} |F_k|^2.$$

Notice that the LHS defines the size of f and the RHS the size of the associated Fourier coefficients. Hence, the Parsevals theorem dictates that the invertible relationship between a periodic function and the associated Fourier coefficients is isometrically isomorphism (roughly speaking invertible and the sizes of a periodic function and the associated Fourier coefficients are the same).

Chapter 5

Signals and Fourier Transform

As shown in the previous chapter, a periodic signal can be decomposed of complex exponentials whose frequencies are integer multiple of that of the periodic signal. However, a signal is in general not periodic. Clearly a non-periodic signal cannot be represented as a Fourier series. Instead, it can be represented as a Fourier transform which is a generalization of Fourier series. Roughly speaking, a nonperiodic signal can be decomposed of complex exponentials of all frequencies.

To see this let f be a periodic function with period T. Then

$$f = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega t}$$

where

$$F_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\omega t}dt.$$

Hence,

$$f = \sum_{k=-\infty}^{\infty} e^{jk\omega t}\frac{\omega}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\omega t}dt.$$

Let

$$g(\phi) = e^{j\phi t}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\phi t}dt.$$

Then

$$f = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} g(k\omega)\omega.$$

For an aperiodic function, $T \to \infty$ and, thus, $\omega \to 0$. Hence,

$$g(\phi) = e^{j\phi t} \int_{-\infty}^{\infty} f(t) e^{-j\phi t} dt$$

and

$$f = \frac{1}{2\pi} \lim_{\omega \to 0} \sum_{k=-\infty}^{\infty} g(k\omega)\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt d\omega.$$

To this end, an aperiodic function f can be expanded with the functions $\{e^{j\omega t} : \omega \in \mathbf{R}\}$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \langle f, e^{j\omega t} \rangle.$$

The above expansion is called the inverse Fourier transform and $F(\cdot)$ is called the Fourier transform.

Similar to the Fourier series, the Fourier transform defines an invertible relationship between a function and the associated Fourier transform. Moreover, the Fourier transform of a function represents the frequency content of the function.

Ex 1: Let

$$f(t) = e^{-t} U(t)$$

where $U(t)$ is the unit step function defined by

$$U(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

Then

$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} e^{-t} U(t) dt = \int_{0}^{\infty} e^{-(1+j\omega)t} dt = \frac{1}{1 + j\omega}.$$

To this end,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + j\omega} e^{j\omega t} d\omega.$$

Notice that the Fourier transform is not defined for $f(t) = e^{-t}$.

Ex 2: Consider the Direc delta function

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

such that

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

Then

$$F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\delta(t-t_0)dt = e^{-j\omega t_0}.$$

Hence

$$\delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}e^{-j\omega t_0}d\omega.$$

Ex 3: Suppose $F(\omega) = \delta(\omega-\omega_0)$. Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\delta(\omega-\omega_0)d\omega = \frac{1}{2\pi}e^{j\omega_0 t}.$$

Hence,

$$\delta(\omega-\omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}e^{j\omega_0 t}dt.$$

Spectra of Signals

The Fourier transform at a frequency represents how much the associated frequency component, $e^{j\omega t}$, the original function has. Hence, similar to Fourier series case, the amplitude and phase spectra as a function of frequency are defined by

$$|F(\omega)| = \sqrt{[Re(F(\omega))]^2 + [Im(F(\omega))]^2}$$

$$\tan\theta(\omega) = \frac{Im(F(\omega))}{Re(F(\omega))}.$$

Notice that, for Fourier series, the spectra could be viewed as a function of frequency $k\omega$.

Similar to Fourier series, the amplitude and phase spectra of a real signal are even and odd functions of ω, respectively. Hence for real signals, it suffices to examine the spectra for $\omega \geq 0$.

Parseval's Theorem

Parseval's Theorem:

$$\int_{-\infty}^{\infty} f(t)\overline{g}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\overline{G}(\omega)d\omega.$$

Proof:

$$\int_{-\infty}^{\infty} f(t)\bar{g}(t)dt = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{G}(\phi)e^{-j\phi t}d\phi\right)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)\bar{G}(\phi)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\phi t}e^{j\omega t}dt\right)d\phi d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)\bar{G}(\phi)\delta(\phi-\omega)d\phi d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\int_{-\infty}^{\infty} \bar{G}(\phi)\delta(\phi-\omega)d\phi d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\bar{G}(\omega)d\omega.$$

If $f = g$, the Parseval's theorem reduces to

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Notice that the LHS defines the size of f and the RHS the size of the associated Fourier transform. Hence, the Parsevals theorem dictates that the invertible relationship between a function and the associated Fourier transform is an isometrically isomorphism.

Convergence of Fourier Transform

Contrary to the Fourier series, the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

may not exist even for a simple function like step function, $f(t) = U(t)$. As mentioned in Example 1, the Fourier transform of $f(t) = e^{-t}$ doesn't exists either. A sufficient condition for the Fourier transform to exist is that $f(t)$ has a finite number of discontinuities over any finite interval and that

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty.$$

Chapter 6

Signals and Laplace Transform

From the convergence consideration of Fourier transform at the end of the previous chapter, the Fourier transform analysis of a signal is limited to a certain class that is not big enough. Hence, the generalization of Fourier transform to a wider class of functions are desirable. Indeed this can be achieved adding exponentially decaying term in the integral over the real line. To this end, consider

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_d(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega$$

where

$$F_d(\sigma + j\omega) = \int_{-\infty}^{\infty} f(t) e^{-\sigma t} e^{-j\omega t} dt.$$

The above expansion is called the inverse Laplace Transform and $F_d(\cdot)$ is called the double-sided Laplace transform. Notice that the Fourier transform is readily recovered if $\sigma = 0$.

Let $s = \sigma + j\omega$. Then we get

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F_d(s) e^{st} ds$$

where

$$F_d(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt.$$

If we are interested in a signal over $[0, \infty)$, the single-sided Laplace transform is obtained as follows:

$$F(s) = \int_{-\infty}^{\infty} f(t) U(t) e^{-st} dt = \int_{0}^{\infty} f(t) e^{-st} dt.$$

Throughout the note, Laplace transform means single-sided Laplace transform unless stated otherwise. If $\sigma > 0$, the Laplace transform is more likely to converge compared to the Fourier transform. Indeed, the convergence is guaranteed if

$$\int_0^\infty |f(t)|e^{-\sigma t}dt = \int_0^\infty |f(t)e^{-st}|dt < \infty.$$

Clearly for $\sigma > 0$, this condition is much weaker than the convergence condition for the Fourier transform. Notice that Laplace transform is impossible for e^{t^2}, e^{t^t}, etc.

Laplace transform for some important functions

1. Unit step (Heavyside) function

$$\mathcal{L}[U(t)] = \int_0^\infty U(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{1}{s}e^{-st}\right]_{t=0}^\infty = \frac{1}{s}.$$

2. Ramp function

$$f(t) = tU(t).$$

$$\mathcal{L}[f(t)] = \int_0^\infty te^{-st}dt = \left[-\frac{t}{s}e^{-st}\right]_{t=0}^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \left[-\frac{1}{s^2}e^{-st}\right]_{t=0}^\infty = \frac{1}{s^2}.$$

3. Unit impulse (Dirac delta) function

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = e^{-s0} = 1.$$

4. Exponential function

$$f(t) = e^{-at}U(t).$$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-at}e^{-st}dt = \left[-\frac{1}{s+a}e^{-(s+a)t}\right]_{t=0}^\infty = \frac{1}{s+a}.$$

5. Sine function

$$f(t) = \sin\omega t U(t).$$

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty \sin\omega t e^{-st}dt = \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j}e^{-st}dt \\ &= \frac{1}{2j}\left[-\frac{1}{s-j\omega}e^{-(s-j\omega)t} + \frac{1}{s+j\omega}e^{-(s+j\omega)t}\right]_{t=0}^\infty \\ &= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}.\end{aligned}$$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$
$\delta(t)$	1
$U(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse Laplace Transform by Partial Fraction Expansion

Ex1 (Distinct Real Factor Case):

$$F(s) = \frac{5}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}.$$

Then

$$A = \lim_{s \to 0} sF(s) = \frac{5}{2},$$

$$B = \lim_{s \to -1} (s+1)F(s) = -5,$$

$$C = \lim_{s \to -2} (s+2)F(s) = \frac{5}{2}.$$

Hence

$$f(t) = AU(t) + Be^{-t}U(t) + Ce^{-2t}U(t).$$

Ex2 (Distinct Complex Factor Case):

$$F(s) = \frac{3}{(s^2+2s+5)s} = \frac{As+B}{s^2+2s+5} + \frac{C}{s} = \frac{A(s+1)+(B-A)}{(s+1)^2+2^2} + \frac{C}{s}$$

$$= \frac{A(s+1)}{(s+1)^2+2^2} + \frac{(B-A)}{2}\frac{2}{(s+1)^2+2^2} + \frac{C}{s}.$$

Then

$$A = \lim_{s \to 0} sF(s) = \frac{3}{5},$$

$$As^2+Bs+Cs^2+2Cs+5C=3 \quad \Rightarrow \quad A=-\frac{3}{5},\ B=-\frac{6}{5}.$$

Hence

$$f(t)=Ae^{-t}\cos 2tU(t)+\frac{(B-A)}{2}e^{-t}\sin 2tU(t)+CU(t).$$

Ex3 (Repeated Factor Case):

$$F(s)=\frac{2}{(s+1)^3 s}=\frac{As^2+Bs+C}{(s+1)^3}+\frac{D}{s}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}+\frac{D}{s}.$$

Then

$$D=\lim_{s\to 0} sF(s)=2,$$

$$C=\lim_{s\to -1}(s+1)^3F(s)=-2,$$

$$B=\lim_{s\to -1}\frac{d}{ds}[(s+1)^3F(s)]=-2,$$

$$A=\frac{1}{2!}\lim_{s\to -1}\frac{d^2}{ds^2}[(s+1)^3F(s)]=-2.$$

Hence,

$$f(t)=Ae^{-t}U(t)+Bte^{-t}U(t)+C\frac{t^2}{2}e^{-t}U(t)+DU(t).$$

Properties of Laplace transform

1. Linearity

$$\mathcal{L}[a_1f(t)+a_2g(t)]=a_1\mathcal{L}[f(t)]+a_2\mathcal{L}[g(t)]$$

Proof: For all $a, b\in \mathbf{R}$,

$$\int_0^\infty e^{-st}(af(t)+bg(t))dt=a\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt.$$

2. Real differentiation

Let

$$\mathcal{L}[f(t)]=F(s).$$

Then

$$\mathcal{L}\left[\frac{df(t)}{dt}\right]=sF(s)-f(0)$$

and

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Proof of the first equality:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

$$= f(\infty)e^{-s\infty} - f(0) + sF(s) = sF(s) - f(0).$$

Notice that taking derivative for $f(t)$ is equal to multiplying s to $F(s)$.

3. Real integration

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}.$$

and

$$\mathcal{L}\left[\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} f(\tau)d\tau dt_{n-1}\cdots dt_1\right] = \frac{F(s)}{s^n}.$$

4. Complex differentiation

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}.$$

5. Real translation

$$\mathcal{L}[f(t-t_0)] = e^{-t_0 s}F(s).$$

Proof: Let $\tau = t - t_0$. $\Rightarrow$ $d\tau = dt$.

$t = 0 \Rightarrow \tau = -t_0$.

$t = \infty \Rightarrow \tau = \infty$.

$$\int_0^\infty f(t-t_0)e^{-st}dt = \int_{-t_0}^\infty f(\tau)e^{-s(\tau+t_0)}d\tau = e^{-t_0 s}\int_{-t_0}^\infty f(\tau)e^{-s\tau}d\tau$$

$$= e^{-t_0 s}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-t_0 s}F(s).$$

Notice that the second last equality follows from $f(\tau) = 0$ for $\tau < 0$.

6. Complex translation

$$\mathcal{L}[e^{at}f(t)] = F(s-a).$$

Proof:

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a).$$

7. Final value theorem for $f(t)$ such that $\lim_{t\to\infty} f(t) < \infty$.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

8. Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

9. Convolution Theorem

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = F(s)G(s)$$

Proof:

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \mathcal{L}\left[\int_{-\infty}^\infty f(\tau)g(t-\tau)d\tau\right]$$

$$= \int_0^\infty e^{-st}\int_{-\infty}^\infty f(\tau)g(t-\tau)d\tau dt \underset{p=t-\tau}{=} \int_{-\tau}^\infty \int_{-\infty}^\infty e^{-s(p+\tau)}f(\tau)g(p)d\tau dp$$

$$= \int_{-\tau}^\infty e^{-s\tau}f(\tau)d\tau \int_{-\infty}^\infty e^{-sp}g(p)dp = \int_0^\infty e^{-s\tau}f(\tau)d\tau \int_0^\infty e^{-sp}g(p)dp$$

$$= F(s)G(s).$$

<u>Connection between Single-Sided and Double-Sided Laplace Transforms</u>

Suppose f be defined on $(-\infty,\infty)$. Then

$$f(t) = f_1(t) + f_2(t)$$

where

$$f_1(t) = f(t)U(t), \quad f_2(t) = f(t)U(-t).$$

Then

$$F_d(s) = \int_{-\infty}^\infty f(t)e^{-st}dt = \int_0^\infty f_1(t)e^{-st}dt + \int_{-\infty}^0 f_2(t)e^{-st}dt$$

$$= \int_0^\infty f_1(t)e^{-st}dt + \int_0^\infty \underbrace{f_2(-t')}_{f'_2(t')}e^{st'}dt' = F_1(s) + F'_2(-s).$$

Part II

Systems

Chapter 7

Preliminaries on Linear Algebra

7.1 Linear Operators

An operator (transformation or mapping) A from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that associates every element in $\mathbf{R}^n$ to an element of $\mathbf{R}^m$.

An operator A from $\mathbf{R}^n$ to $\mathbf{R}^m$ is said to be linear if

$$A(\alpha x + \beta \hat{x}) = \alpha A x + \beta A \hat{x}, \quad \forall x, \hat{x} \in \mathbf{R}^n.$$

Terminologies:

1. Null Space (Kernel):

$$\mathcal{N}(A) = \{u \in \mathbf{R}^n : Au = 0\}$$

   The dimension of null space is called the nullity.

2. Range Space (Image):

$$\mathcal{R}(A) = \{v \in \mathbf{R}^m : v = Au, u \in \mathbf{R}^n\} = A\mathbf{R}^n$$

   The dimension of range space is called the rank.

Matrix Representation of Linear Operators

Let $\{u_j\}_{j=1}^{n}$ be the basis for $\mathbf{R}^n$. Then

$$x = \sum_{j=1}^{n} \xi_j u_j.$$

By linearity of $\boldsymbol{A}$,

$$\boldsymbol{A}x = \boldsymbol{A}\sum_{j=1}^{n} \xi_j u_j = \sum_{j=1}^{n} \xi_j \boldsymbol{A} u_j.$$

Let $\{v_i\}_{i=1}^{m}$ be the basis for $\mathbf{R}^m$. Then

$$\boldsymbol{A}u_j = \sum_{i=1}^{m} a_{ij} v_i$$

$$\Downarrow$$

$$\sum_{i=1}^{m} \eta_i v_i = y = \boldsymbol{A}x = \sum_{j=1}^{n} \xi_j \boldsymbol{A}u_j = \sum_{j=1}^{n} \xi_j \sum_{i=1}^{m} a_{ij} v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \xi_j \right) v_i.$$

Hence,

$$\eta = \boldsymbol{A}\xi$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Theorem: Let $\{u_j\}_{j=1}^{n}$ and $\{v_i\}_{i=1}^{m}$ be the bases for $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then, w.r.t. these bases, $\boldsymbol{A}$ is represented by the $m \times n$ matrix.

Change of Basis

Let $\{u_k\}_{k=1}^{n}$ and $\{\tilde{u}_i\}_{i=1}^{n}$ be two bases for $\mathbf{R}^n$ and $\{v_k\}_{k=1}^{m}$ and $\{\tilde{v}_i\}_{i=1}^{m}$ two bases for $\mathbf{R}^m$. Then

$$\tilde{u}_i = \sum_{k=1}^{n} p_{ki} u_k$$

$$\Downarrow$$

$$\sum_{k=1}^{n} \xi_k u_k = x = \sum_{i=1}^{n} \tilde{\xi}_i \tilde{u}_i = \sum_{i=1}^{n} \tilde{\xi}_i \left(\sum_{k=1}^{n} p_{ki} u_k \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} p_{ki} \tilde{\xi}_i \right) u_k.$$

$$\Downarrow$$

$$\xi = P\tilde{\xi}$$

Notice that the ith column of P is the representation of $\tilde{u}_i$ w.r.t $\{u_j\}$.

Similarly,

$$\tilde{\eta} = Q\eta$$

Notice that the ith column of Q is the representation of v_i w.r.t $\{\tilde{v}_j\}$.

Let $y = Ax \Rightarrow \eta = A\xi \Rightarrow$

$$\tilde{\eta} = QA\xi = QAP\tilde{\xi}$$

$$\Downarrow$$

the representation of linear operator w.r.t. $\{\tilde{u}_i\}$ and $\{\tilde{v}_i\}$ is

$$\tilde{A} = QAP$$

Special Case: $V = U$ and use same basis for both domain and range. Then

$$\xi = P\tilde{\xi} = PQ\xi \quad \Rightarrow \quad PQ = I \quad \Rightarrow \quad Q = P^{-1} \quad \Rightarrow \quad \tilde{A} = P^{-1}AP$$

Such transformation from A to $\tilde{A}$ is called similarity transformation.

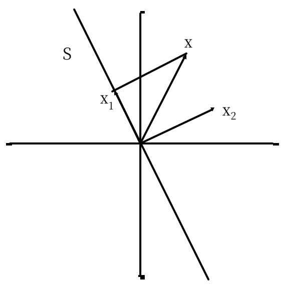
Orthogonal Decomposition

Let S be a subspace of $\mathbf{R}^n$. Then the orthogonal complement of S is defined as

$$S^{\perp} := \{x \in \mathbf{R}^n : \langle x, y \rangle = 0, \ \forall y \in S\}.$$

Fact (Orthogonal Decomposition): $\mathbf{R}^n = S \oplus S^{\perp}$.

Proof: Suppose $x \in \mathbf{R}^n$. Let x_1 be the projection of x on S. Then $x_2 = x - x_1$ is orthogonal to S and thus $x_2 \in S^{\perp}$.



$$y \in [\mathcal{R}(A)]^{\perp} \quad \Leftrightarrow \quad y^* z = 0 \; \forall z \in \mathcal{R}(A)$$

$$\Leftrightarrow \quad (A^* y)^* x = y^* A x = 0 \; \forall x \in \mathbf{R}^n \quad \Leftrightarrow \quad A^* y = 0 \quad \Leftrightarrow \quad y \in \mathcal{N}(A^*).$$

Hence, $\mathcal{N}(A^*)$ is the orthogonal complement of $\mathcal{R}(A)$ and thus

$$\mathbf{R}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^*).$$

Similarly,

$$\mathbf{R}^n = \mathcal{R}(A^*) \oplus \mathcal{N}(A).$$

Eigenvalues and Eigenvectors

Def: $\lambda \in \mathbf{C}$ is called an eigenvalue of A if $\exists$ right (left) eigenvector $x(y) \neq 0$ such that $Ax = \lambda x$ $(y^* A = \lambda y^*)$.

Fact: λ is an eigenvalue of A iff it is a solution of the characteristic polynomial

$$\chi_A(\lambda) = det(\lambda I - A) = 0.$$

The eigenvector x is a nonzero vector in $\mathcal{N}(\lambda I - A)$.

Theorem: Let $\lambda_1, \cdots, \lambda_n$ be the distinct eigenvalues of A and v_i be an eigenvector associated with λ_i. Then $\{v_i\}_{i=1}^n$ is linearly independent.

Proof: Suppose the contrary. $\exists\, a_i$'s (not all zero) such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

WLOG, we assume $a_1 \neq 0$. Consider

$$(A - \lambda_2 I) \cdots (A - \lambda_n I) \left(\sum_{i=1}^{n} a_i v_i \right) = 0.$$

Notice that

$$(A - \lambda_j I) v_i = (\lambda_i - \lambda_j) v_i \quad \text{if } j \neq i$$

and

$$(A - \lambda_i I) v_i = 0.$$

Hence,

$$a_1 (\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) v_1 = 0.$$

Since λ_i's are distinct, this implies $a_1 = 0$ (contradiction!).

Def.: A matrix is simple if it has n linearly independent eigenvectors.

Corollary: If eigenvalues of A are all distinct, A is simple.

Remark: There exist simple matrices whose eigenvalues of A are not all distinct. (Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Let A be simple. Then notice that

$$AV = V\Lambda$$

where

$$V = [v_1 \ \cdots \ v_n] \quad \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}.$$

Since V is nonsingular, we have

$$V^{-1}AV = \Lambda.$$

Note that Λ is the representation of A in terms of its eigenvectors.

Fact: If A is simple, A can be diagonalised by similarity transform.

Positive Definite Hermitian Square Matrix

Def.: A is Hermitian iff $A = A^*$.

Fact: Let A be Hermitian.

1. x^*Ax is real.

2. eigenvalues of A are all real.

3. n eigenvectors exist and are all orthogonal.

Proof: 1) $(x^*Ax)^* = x^*A^*x = x^*Ax$.

2) Let λ be an eigenvalue and v be the corresponding eigenvector. Then $v^*Av = \lambda v^*v$. Note that LHS is real, and v^*v is real and > 0.

3) (Proof of orthogonality) For multiple eigenvalues, we can always choose mutually orthogonal eigenvectors. Suppose $Au = \lambda u$ and $Av = \mu v$ with $\lambda \neq \mu$. Note that $u^*A = \lambda u^*$. Hence

$$u^*Av = \lambda u^*v \quad \text{and} \quad u^*Av = \mu u^*v$$

$\Rightarrow \lambda u^*v = \mu u^*v \Rightarrow u^*v = 0$.

Def.: A is positive semidefinite (PSD) if $x^*Ax \geq 0$ for all x.

Def.: A is positive definite (PD) if $x^*Ax > 0$ for all $x \neq 0$.

Fact: TFAE

1. A is PSD (PD).

2. all its eigenvalues are nonnegative (positive).

Proof: ($1 \Rightarrow 2$) Let λ_i be an eigenvalue and v_i be the corresponding unit eigenvector. Then

$$Av_i = \lambda_i v_i \quad \Rightarrow \quad 0 \leq (<) v_i^* A v_i = \lambda_i v_i^* v_i = \lambda_i.$$

($2 \Rightarrow 1$) $\{v_i\}$ orthonormal eigenvectors

$$Ax = A(a_1 v_1 + \cdots + a_n v_n) = a_1 A v_1 + \cdots + a_n A v_n = a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n$$

$$\Downarrow$$

$$x^* A x = (a_1 v_1^* + \cdots + a_n v_n^*)(a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n) = a_1^2 \lambda_1 + \cdots + a_n^n \lambda_n \geq (>) 0.$$

Functions of Matrices

Let A be a square matrix and $p(t)$ be a polynomial:

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

Then the matrix polynomial is defined as

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Cayley Hamilton Theorem: Let $\chi_A(\lambda) = det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be the characteristic polynomial of A. Then

$$\chi_A(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n = 0$$

Proof for simple A: Let v_i be an eigenvector of A associated with eigenvalue λ_i. Then

$$\chi_A(A) V = V diag\{\chi_A(\lambda_1), \cdots, \chi_A(\lambda_n)\} = 0$$

where $V = [v_1 \ \cdots \ v_n]$. Since V is nonsingular, $\chi_A(A) = 0$.

Corollary: A^k, $k \geq n$, is a linear combination of $I, A, \cdots, A^{n-1}$.

Matrix Exponential

Consider the Taylor series expansion of the exponential function e^{at}:

$$e^{at} = 1 + at + \frac{a^2}{2!}t^2 + \cdots + \frac{a^n}{n!}t^n + \cdots.$$

Now the matrix exponential e^{At} is defined as

$$e^{At} = \boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2}{2!}t^2 + \cdots + \frac{\boldsymbol{A}^n}{n!}t^n + \cdots.$$

Fact: Properties of Matrix Exponential e^{At}

1.
$$\frac{d}{dt}e^{At} = \boldsymbol{A}e^{At} = e^{At}\boldsymbol{A}.$$

2.
$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}.$$

3. e^{At} is nonsingular and
$$[e^{At}]^{-1} = e^{-At}.$$

4. For nonsingular $\boldsymbol{P}$,
$$e^{PAP^{-1}t} = \boldsymbol{P}e^{At}\boldsymbol{P}^{-1}.$$

5.
$$e^{At} = \mathcal{L}^{-1}(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \mathcal{L}^{-1}(\boldsymbol{I}s^{-1} + \boldsymbol{A}s^{-2} + \boldsymbol{A}^2s^{-3} + \cdots).$$

6. The matrix exponential can be written as a finite order polynomial
$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^k.$$

Proof: 1) and 4) are obvious from the series representation of e^{At}.
2) Consider $e^{At}x_0$. Then

$$\frac{d}{dt}(e^{At}x_0) = \boldsymbol{A}e^{At}x_0.$$

Hence $e^{At}x_0$ is the solution to

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x_0.$$

To this end, for all x_0,

$$e^{A(t_1+t_2)}x_0 = x(t_1 + t_2) = e^{At_1}x(t_2) = e^{At_1}e^{At_2}x_0.$$

3) 2) $\Rightarrow e^{At}e^{-At} = I \Rightarrow [e^{At}]^{-1} = e^{-At} \Rightarrow e^{At}$ nonsingular.

5) Taking LT's of $\dot{x}(t) = Ax(t)$ with $x(0) = x_0$ and $x(t) = e^{At}x_0$, we get

$$sX(s) = AX(s) + x_0 \quad \Rightarrow \quad X(s) = (sI - A)^{-1}x_0$$

and

$$X(s) = \mathcal{L}e^{At} \cdot x_0,$$

respectively. Hence,

$$(sI - A)^{-1} = \mathcal{L}e^{At} = \mathcal{L}\left(\sum_{k=0}^{\infty}\frac{t^k A^k}{k!}\right) = \sum_{k=0}^{\infty} A^k s^{-k-1}.$$

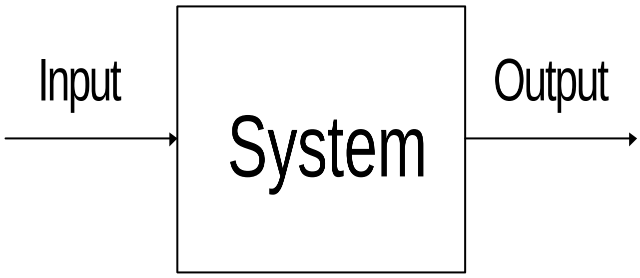
6) follows from the Cayley-Hamilton Theorem.

Notice that the matrix exponential can be computed using Fact 5) & 6).

Chapter 8

Introduction to Systems

A system is a signal processor that processes the input signal and gives the output signal.



$$y = \mathcal{S}u$$

A system is mathematically described by an equation between input and output.

A system is linear if for any scalars $a_1, a_2 \in \mathcal{F}$ and signals u_1, u_2,

$$\mathcal{S}(a_1u_1 + a_2u_2) = a_1\mathcal{S}u_1 + a_2\mathcal{S}u_2.$$

Notice that $\mathcal{S}0 = 0 \times \mathcal{S}1 = 0$.

A system is time invariant if for any τ

$$y(\cdot) = \mathcal{S}u(\cdot), \quad z(\cdot) = \mathcal{S}u(\cdot - \tau),$$

$$\Downarrow$$

$$z(\cdot) = y(\cdot - \tau).$$

A system is causal (physical, nonanticipative) if the output $y(t)$ depends only on the past and the current input $u(\tau), \tau \leq t$. Notice that any physically meaningful system must be causal.

A causal system is instantaneous (static, stationary, memoryless) if the output $y(t)$ depends only on the current input $u(t)$. Otherwise a causal system is called dynamic. Usually a static system is mathematically described by an algebraic equation between input and output whereas a dynamic system by a differential equation.

Chapter 9

Representation of Linear Dynamic Systems

9.1 Differential Equation Models

Differential Equation Model

A linear dynamic physical system is modeled by a linear differential equation (which may be an approximation of a nonlinear dynamic system through linearization of a nonlinear differential equation):

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u.$$

Notice that the largest order of differentiation of LHS is greater ($b_0 = 0$) than or equal ($b_0 \neq 0$) to that of RHS and, thus, the system is causal. The solution consists of the homogeneous part y_h and the nonhomogenous part y_n. Clearly the first depends on the initial conditions:

$$y(t_0) = y_0,\ \dot{y}(t_0) = \dot{y}_0,\ \cdots,\ y^{(n-1)}(0) = y_0^{(n-1)},$$

whereas the second on the forcing function (RHS of the equation) that is the input u. Hence y_h (y_n) represents the effects of initial condition (input) on the output.

State Space Model

Consider

$$\xi^{(n)} + a_1 \xi^{(n-1)} + \cdots + a_{n-1}\dot{\xi} + a_n \xi = u. \quad (*)$$

Then by linearity

$$y = b_0\xi^{(n)} + b_1\xi^{(n-1)} + \cdots + b_{n-1}\dot{\xi} + b_n\xi.$$

Moreover using $(*)$,

$$y = \beta_1^0\xi^{(n-1)} + \cdots + \beta_{n-1}^0\dot{\xi} + \beta_n^0\xi + b_0 u$$

where

$$\beta_i^0 = b_i - b_0 a_i.$$

Now from the initial condition $y(t_0) = y_0$,

$$y_0 = y(t_0) = \beta_1^0\xi^{(n-1)}(t_0) + \cdots + \beta_{n-1}^0\dot{\xi}(t_0) + \beta_n^0\xi(t_0) + b_0 u(t_0).$$

Similarly, for $i = 0, \cdots, n-1$,

$$y_0^{(i)} = y^{(i)}(t_0) = \beta_1^0\xi^{(n+i-1)}(t_0) + \cdots + \beta_{n-1}^0\xi^{(i+1)}(t_0) + \beta_n^0\xi^{(i)}(t_0) + b_0 u^{(i)}(t_0).$$

and, using $(*)$ successively, $(*)$ for i,

$$y_0^{(i)} - B_0^i u^{(i)}(t_0) - \cdots - B_i^i u(t_0) = \beta_1^i\xi^{(n-1)}(t_0) + \cdots + \beta_{n-1}^i\dot{\xi}(t_0) + \beta_n^i\xi(t_0). \quad (**)$$

Notice that we have n equations and n unknowns of $\xi^{(i)}(t_0)$. Hence, the differential equation in the previous section is equivalent to the equation:

$$\xi^{(n)} + a_1\xi^{(n-1)} + \cdots + a_{n-1}\dot{\xi} + a_n\xi = u,$$

$$y = \beta_1^0\xi^{(n-1)} + \cdots + \beta_{n-1}^0\dot{\xi} + \beta_n^0\xi + b_0 u$$

with the initial condition computed from the equations $(**)$.

Remark 1: If RHS of the differential equation in the previous chapter is u, then $\xi = y$.

Remark 2: If $b_0 = 0$, then $\beta_1^0 = b_1, \cdots, \beta_n^0 = b_n$.

Let

$$x_1 = \xi, \; x_2 = \dot{\xi}, \; \cdots, \; x_{n-1} = \xi^{(n-2)}, \; x_n = \xi^{(n-1)}.$$

Then

$$\dot{x}_1 = \dot{\xi} = x_2$$

$$\dot{x}_2 = \ddot{\xi} = x_3$$

$$\vdots$$
$$\dot{x}_{n-1} = \xi^{(n-1)} = x_n$$
$$\dot{x}_n = \xi^{(n)} = -a_1\xi^{(n-1)} - \cdots - a_{n-1}\dot{\xi} - a_n\xi + u$$
$$= -a_1 x_n - \cdots - a_{n-1}x_2 - a_n x_1 + u.$$

Hence the differential equation can be rewritten as the so-called controller canonical form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [\beta_n^0 \; \beta_{n-1}^0 \; \cdots \; \beta_1^0]x + b_0 u.$$

In general, a linear time invariant system is described by

$$\dot{x} = \boldsymbol{A}x + bu \quad \text{State DE}$$

$$y = c^T x + du \quad \text{Readout Map}$$

Notice that the system is completely characterized by the matrix $[\boldsymbol{A}, b, c, d]$.

Fact: the closed form solution of the state space equation is

$$x(t) = \underbrace{e^{At}x_0}_{x_h} + \underbrace{\int_0^t e^{A(t-\tau)}bu(\tau)d\tau}_{x_n}.$$

Proof: At $t = 0$,

$$x(0) = x_0$$

Moreover,

$$\dot{x}(t) = \boldsymbol{A}e^{At}x_0 + bu(t) + \int_0^t \boldsymbol{A}e^{A(t-\tau)}bu(\tau)d\tau = \boldsymbol{A}x(t) + bu(t).$$

Hence,

$$y(t) = \underbrace{c^T e^{At}x_0}_{y_h} + \underbrace{\int_0^t c^T e^{A(t-\tau)}bu(\tau)d\tau + du(t)}_{y_n}.$$

<u>State</u>

Given a time instant t, the state of the system is the minimal information that are necessary to calculate the future response.

For ODE's, the concept of the state is the same as that of the initial condition.

$$\Downarrow$$

$$\text{State} = x(t)$$

Consider the change of coordinate of the state space such that $\bar{x} = \boldsymbol{P}x$. Then

$$\dot{x} = \boldsymbol{A}x + bu, \quad y = c^T x + du$$

$$\Downarrow$$

$$\dot{\bar{x}} = \bar{\boldsymbol{A}}\bar{x} + \bar{b}u, \quad y = \bar{c}^T \bar{x} + \bar{d}u$$

where

$$\bar{\boldsymbol{A}} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} \quad \bar{b} = \boldsymbol{P}b \quad \bar{c}^T = c^T \boldsymbol{P}^{-1} \quad \bar{d} = d.$$

Hence, two systems represented by $[\boldsymbol{A}, b, c, d]$ and $[\bar{\boldsymbol{A}}, \bar{b}, \bar{c}, \bar{d}]$ are equivalent because the only difference is the coordinate system of the state space.

Finally for an input such that $x(-\infty) = 0$,

$$0 = x(-\infty) = \lim_{t \to \infty} \left[e^{-\boldsymbol{A}t} x_0 + \int_0^{-t} e^{-\boldsymbol{A}t} e^{-\boldsymbol{A}\tau} bu(\tau) d\tau \right]$$

$$= \lim_{t \to \infty} \left[e^{-\boldsymbol{A}t} \left(x_0 - \int_{-t}^{0} e^{-\boldsymbol{A}\tau} bu(\tau) d\tau \right) \right].$$

If $\lim_{t \to \infty} e^{-\boldsymbol{A}t} = \infty$,

$$x_0 = \int_{-\infty}^{0} e^{-\boldsymbol{A}\tau} bu(\tau) d\tau.$$

Hence the initial condition can be viewed as a condensed core memory of the past. Notice that

$$x(t) = e^{\boldsymbol{A}t} x_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)} bu(\tau) d\tau$$

$$= \int_{-\infty}^{0} e^{\boldsymbol{A}(t-\tau)} bu(\tau) d\tau + \int_0^t e^{\boldsymbol{A}(t-\tau)} bu(\tau) d\tau = \int_{-\infty}^{t} e^{\boldsymbol{A}(t-\tau)} bu(\tau) d\tau.$$

9.2 Input-Output Models

An input-output model describes the effects of input on the output only. Hence, the initial condition is assumed to be the zero steady state. Thus this assumption will be adopted anywhere an input-output model is considered.

Laplace Domain Model: Transfer Function

Under zero initial condition assumption, the Laplace transform of the original nth order differential equation is

$$s^n \boldsymbol{Y}(s) + a_1 s^{n-1} \boldsymbol{Y}(s) + \cdots + a_n \boldsymbol{Y}(s) = b_0 s^n \boldsymbol{U}(s) + b_1 s^{n-1} \boldsymbol{U}(s) + \cdots + b_n \boldsymbol{U}(s).$$

Hence

$$G(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{U}(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \quad \Rightarrow \quad \boldsymbol{Y}(s) = G(s)\boldsymbol{U}(s).$$

$G(s)$ is called transfer function of the system. The denominator polynomial is called the characteristic polynomial and its solutions are called the poles. On the other hand, the solutions of the numerator polynomial is called he zeros.

b_0: finite $\Rightarrow$ system is proper.

$b_0 = 0 \Rightarrow$ system is strictly proper.

On the other hand, the Laplace transform of the state space equation is

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + b\boldsymbol{U}(s)$$

$$\boldsymbol{Y}(s) = c^T \boldsymbol{X}(s) + d\boldsymbol{U}(s)$$

$$\Downarrow$$

$$\boldsymbol{Y}(s) = [c^T(s\boldsymbol{I} - \boldsymbol{A})^{-1} b + d]\boldsymbol{U}(s)$$

Hence the transfer function associated with the state equation is

$$G(s) = c^T(s\boldsymbol{I} - \boldsymbol{A})^{-1} b + d.$$

d: finite $\Rightarrow$ system is proper.

$d = 0 \Rightarrow$ system is strictly proper.

Suppose $[\boldsymbol{A}, b, c, d]$ and $[\bar{\boldsymbol{A}}, \bar{b}, \bar{c}, \bar{d}]$ are equivalent. Then

$$\bar{c}^T(s\boldsymbol{I} - \bar{\boldsymbol{A}})^{-1}\bar{b} + \bar{d} = c^T \boldsymbol{P}^{-1}(s\boldsymbol{I} - \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1})^{-1}\boldsymbol{P}b + d$$

$$= c^T P^{-1}[P(sI - A)P^{-1}]^{-1}Pb + d = c^T(sI - A)^{-1}b + d.$$

Hence two equivalent state equations result in the same transfer function and thus are the two different representation of a system.

Fact: state space representation of an I/O description is not unique.

Realisation problem: Given G, what is the state space realisation $[A, b, c, d]$ whose transfer function is G?

Clearly the controller form of realisation can be obtained transforming the transfer function into differential equation model. Different realisation can be obtained by changing the coordinate of the state space.

Finally notice that if

$$x_0 = \int_{-\infty}^{0} e^{-A\tau} bu(\tau)d\tau,$$

the double sided Laplace transform needs to be used. In that case, notice that U and Y are different but G remains the same.

<u>Time Domain Model: Convolution</u>

Impulse Response ($g(t)$): $y(t)$ when $x_0 = 0$ and $u = \delta$

$$g(t) = \int_0^t e^{-A(t-\tau)} b\delta(\tau)d\tau + d\delta(t) = c^T e^{At} b + d\delta(t) \quad \forall t \geq 0$$

$$\Downarrow$$

$$y(t) = \int_0^t c^T e^{A(t-\tau)} bu(\tau)d\tau + du(t)$$

$$= \int_0^t g(t-\tau)u(\tau)d\tau = g * u(t) \quad \text{Convolution}$$

Taking Laplace transform,

$$Y(s) = G(s)U(s).$$

Hence, transfer function is the Laplace transform of the impulse response. Notice that the Laplace transform of the delta function is 1.

Suppose

$$x_0 = \int_{-\infty}^{0} e^{-A\tau} bu(\tau)d\tau.$$

Then

$$y(t) = \int_{-\infty}^{t} c^T e^{A(t-\tau)} bu(\tau)d\tau + du(t).$$

Now the impulse Response $g(t)$ is $y(t)$ when $u(t) = \delta(t)$:

$$g(t) = \int_{-\infty}^{t} c^T e^{A(t-\tau)} b\delta(\tau)d\tau + d\delta(t) = \begin{cases} c^T e^{At} b + d\delta(t) & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Hence, the convolution is

$$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau.$$

Taking double-sided Laplace transform,

$$\boldsymbol{Y}_d(s) = \boldsymbol{G}_d(s)\boldsymbol{U}_d(s) = \boldsymbol{G}(s)\boldsymbol{U}_d(s).$$

Notice that transfer function remains the same.

Fact: For all $-\infty < t < \infty$,

$$G(s) = \frac{\text{output subject to the input } e^{st}}{e^{st}}.$$

Proof:

$$y(t) = \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau = \int_{0}^{\infty} g(\tau')u(t-\tau')d\tau'.$$

If $u = e^{st}$,

$$y(t) = \int_{0}^{\infty} g(\tau')e^{s(t-\tau')}d\tau' = e^{st}\int_{0}^{\infty} g(\tau')e^{-s\tau'}d\tau' = e^{st}G(s).$$

Corollary: For all $-\infty < t < \infty$,

$$G(j\omega) = \frac{\text{output subject to the input } e^{j\omega t}}{e^{j\omega t}}.$$

Chapter 10

Dynamic Responses to Typical Inputs

10.1 Step Response

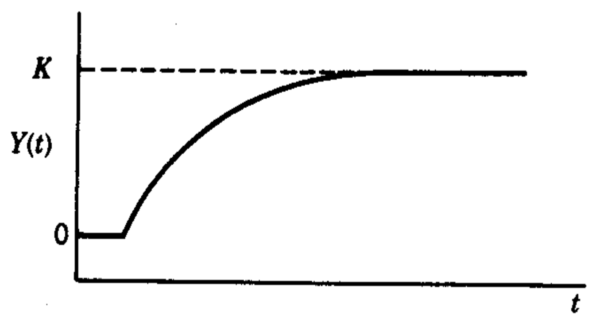
Step Input: $u(t) = U(t) \Rightarrow U(s) = \frac{1}{s}$.

First Order Systems: $G_p(s) = \frac{K}{\tau s+1}$

$$Y(s) = G_p(s)U(s) = \frac{K}{s(\tau s+1)} = \frac{K}{s} - \frac{K\tau}{\tau s+1}.$$

Taking the inverse Laplace transform:

$$y(t) = K(1 - e^{-\frac{t}{\tau}}).$$



Remark 1: $y(t) \to K$ as $t \to \infty$. Hence the long term behavior is determined by K that is called steady state gain.

Remark 2: $\frac{dy}{dt}(0) = \frac{K}{\tau}$. For given K, the short term response is determined by τ that is called time constant. The smaller τ results in faster convergence to the value of K.

<u>Second Order Systems:</u> $G_p(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$

$$Y(s) = G_p(s)U(s) = \frac{K}{s(\tau^2 s^2 + 2\xi\tau s + 1)}.$$

From

$$\tau^2 s^2 + 2\xi\tau s + 1 = 0,$$

the poles of the plant are

$$p_{1,2} = -\frac{\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau}.$$

Case 1: Overdamped response ($\xi > 1 \Rightarrow$ two distinct real poles)

Taking the inverse Laplace transform:

$$y(t) = K\left[1 - e^{-\frac{\xi}{\tau}t}\left(\cosh\frac{\sqrt{\xi^2-1}}{\tau}t + \frac{\xi}{\sqrt{\xi^2-1}}\sinh\frac{\sqrt{\xi^2-1}}{\tau}t\right)\right].$$

Case 2: Critically damped response ($\xi = 1 \Rightarrow$ double real poles)

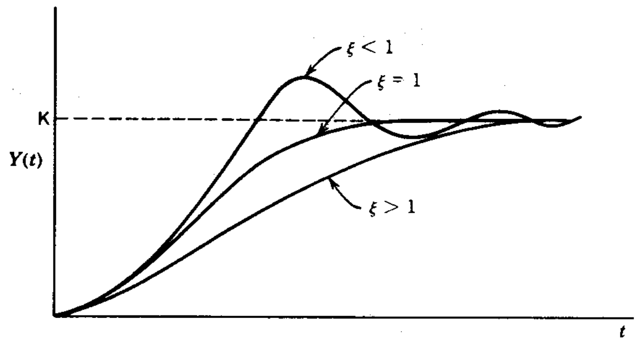
Taking the inverse Laplace transform:

$$y(t) = K\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-\frac{t}{\tau}}\right].$$

Case 3: Underdamped response ($\xi < 1 \Rightarrow$ two distinct complex conjugate poles)

Taking the inverse Laplace transform:

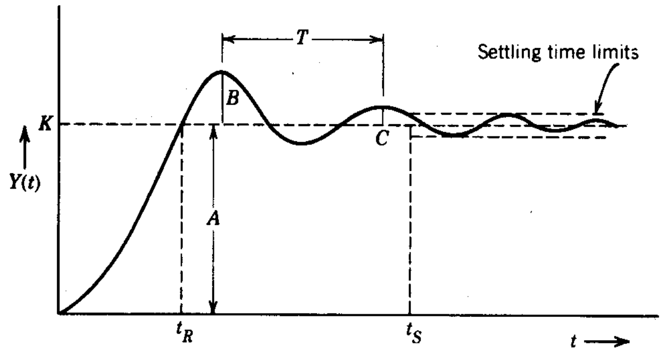
$$y(t) = K\left[1 - \frac{1}{\sqrt{1-\xi^2}}e^{-\frac{\xi}{\tau}t}\sin\left(\frac{\sqrt{1-\xi^2}}{\tau}t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)\right].$$



Remark 1: $y(t) \to K$ as $t \to \infty$ in any case and K is called the steady state gain.

Remark 2: For given K, the short term response now depends on both the damping ratio ξ and the time constant τ.

Characteristics of underdamped response:



- Overshoot: $\frac{B}{A} = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$
- Decay ratio: $\frac{C}{B} = e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$
- Rise time (t_R): the time at which the response first reaches to the final value K

- Settling time (t_S): the smallest time after which the response lies within some prescribed limits of the final value K
- Period of oscillation: $T = \frac{2\pi\tau}{\sqrt{1-\xi^2}}$
- Natural period of oscillation (T at $\xi = 0$): $T_n = 2\pi\tau$

10.2 Periodic Input Response

Fact: the response of a linear time-invariant system subject to a periodic input is also periodic with the same period as that of the input.

Proof: Let u be periodic with period T. Then

$$u(t) = \sum_{-\infty}^{\infty} U_k e^{jk\omega t}.$$

Now from the corollary in the previous chapter,

$$y(t) = \sum_{-\infty}^{\infty} U_k G(jk\omega) e^{jk\omega t}.$$

Notice that $Y_k = G(jk\omega)U_k$ and thus

$$|Y_k| = |G(jk\omega)||U_k|$$

$$\angle(Y_k) = \angle(G(jk\omega)) + \angle(U_k).$$

Ex: Consider $u(t) = u_0 \sin \omega t$. Notice that

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Hence

$$U_k = \begin{cases} -\frac{u_0}{2j} & \text{if } k = -1 \\ \frac{u_0}{2j} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore

$$y(t) = \frac{u_0 G(j\omega)}{2j} e^{j\omega t} - \frac{u_0 G(-j\omega)}{2j} e^{-j\omega t}.$$

Let

$$\theta = \angle G(j\omega).$$

Then since $G(j\omega) = \overline{G(-j\omega)}$,

$$y(t) = u_0|G(j\omega)|\frac{e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t}}{2j} = u_0|G(j\omega)|\sin(\omega t + \theta) = y_0 \sin(\omega t + \theta)$$

where $y_0 = u_0|G(j\omega)|$. Hence the output is also sine wave with the same period although the amplitude has changed and phase angle has shifted.

10.3 Frequency Response

Given an input signal u,

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)e^{j\omega t}d\omega.$$

Now from the corollary in the previous chapter,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)G(j\omega)e^{j\omega t}d\omega.$$

Hence $Y(\omega) = G(j\omega)U(\omega)$ where $G(j\omega)$ is called the frequency response function. To this end, the input signal is decomposed into different frequency components through Fourier transform, a frequency component of the input with frequency ω is adjusted by the system to give the frequency component of the output with the same frequency, and the output signal is obtained from the frequency components of the output through inverse Fourier transform.

Notice that

$$|Y(\omega)| = |G(j\omega)||U(\omega)|$$
$$\angle(Y(\omega)) = \angle(G(j\omega)) + \angle(U(\omega)).$$

Amplitude ratio (AR):

$$AR(\omega) = \frac{|Y(\omega)|}{|U(\omega)|} = |G(j\omega)|$$

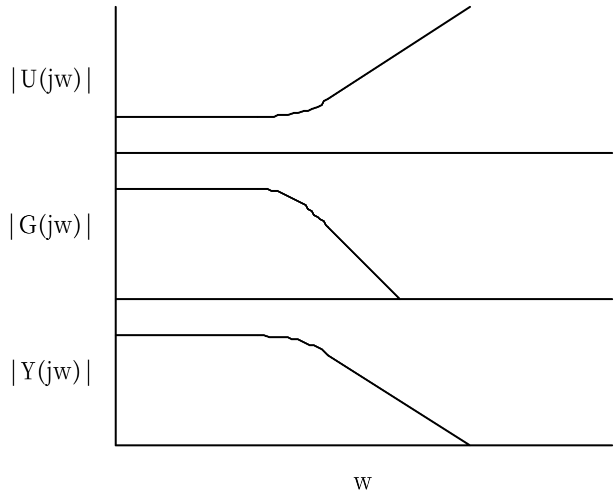
Magnitude ratio (MR):

$$MR(\omega) = \frac{AR(\omega)}{K}$$

where K is the steady state gain of the plant.

Phase angle: $\theta(\omega) = \angle(Y(\omega)) - \angle(U(\omega)) = \angle(G(j\omega))$

Notice that $U(\omega)$ and $Y(\omega)$ represent the content of ω frequency component in input and output, respectively.



Hence if $AR(\omega) \underset{(<)}{>} 1$, ω frequency component of input signal is amplified (attenuated).

Example: Consider

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}.$$

Then

$$AR(\omega) = \left|\frac{K}{j\omega\tau + 1}\right| = \frac{K}{\sqrt{1 + \tau^2\omega^2}}$$

$$\theta(\omega) = \angle G(j\omega) = \angle\frac{K}{j\omega\tau + 1} = \angle K - \angle(1 + j\omega\tau) = -\arctan(\omega\tau).$$

Question: How do $AR(\omega)$ and $\theta(\omega)$ behave as ω changes?

Graphical Representation of AR and θ:

- Bode plot: $\log AR$ vs $\log\omega$ and θ vs $\log\omega$
- Nyquist plot: $Re[G(j\omega)]$ vs $Im[G(j\omega)]$

10.3.1 Bode Plot

<u>First order system:</u> $G_p(s) = \frac{K}{\tau s+1}$

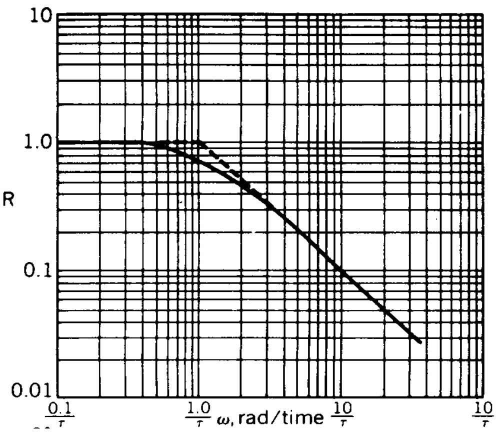
$$AR = \frac{K}{\sqrt{\omega^2\tau^2+1}}, \quad MR = \frac{1}{\sqrt{\omega^2\tau^2+1}}, \quad \theta = -\arctan(\omega\tau)$$

Step 1: Asymptotes

As $\omega \to 0$, $MR \to 1 \Rightarrow \log MR \to 0$

As $\omega \to \infty$, $MR \to \frac{1}{\omega\tau} \Rightarrow \log MR \to \log\frac{1}{\tau} - \log\omega$ (= 0 at $\omega_c = \frac{1}{\tau}$ and slope = -1)

Step 2: $MR(\omega_c) = \frac{1}{\sqrt{2}}$

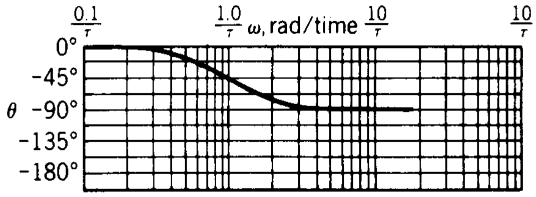


Step 3:

As $\omega \to 0$, $\theta \to 0$

As $\omega \to \infty$, $\theta \to -\frac{\pi}{2}$

$\theta(\omega_c) = -\frac{\pi}{4}$



Second order system: $G_p(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$

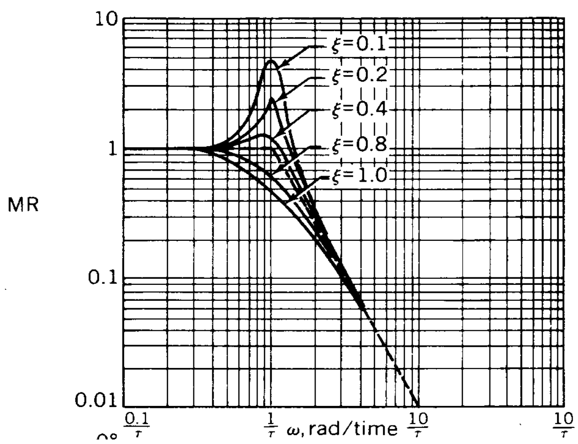
$$AR = \frac{K}{\sqrt{(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2\omega^2}}, \quad MR = \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2\omega^2}},$$

$$\theta = -\arctan\left(\frac{2\xi\omega\tau}{1-\tau^2\omega^2}\right)$$

Step 1: Asymptotes

As $\omega \to 0$, $MR \to 1 \Rightarrow \log MR \to 0$

As $\omega \to \infty$, $MR \to \frac{1}{\omega^2\tau^2} \Rightarrow \log MR \to \log\frac{1}{\tau^2} - 2\log\omega$ (= 0 at $\omega_c = \frac{1}{\tau}$ and slope = -2)



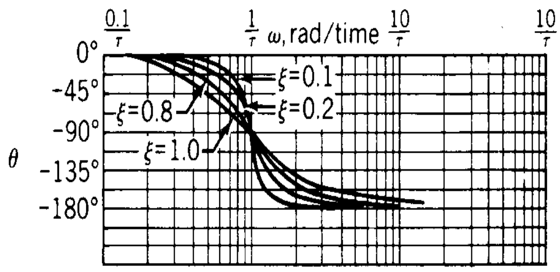
Step 2: $\boldsymbol{MR}_{max}$?

$$\frac{d}{dt}\left(\frac{1}{\sqrt{f(t)}}\right) = -\frac{f'(t)}{2\sqrt{f^3(t)}} = 0 \quad \Rightarrow \quad f'(t) = 0$$

$$\frac{d\boldsymbol{MR}}{d\omega} = 0 \quad \Rightarrow \quad \frac{d[(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2\omega^2]}{d\omega} = 0$$

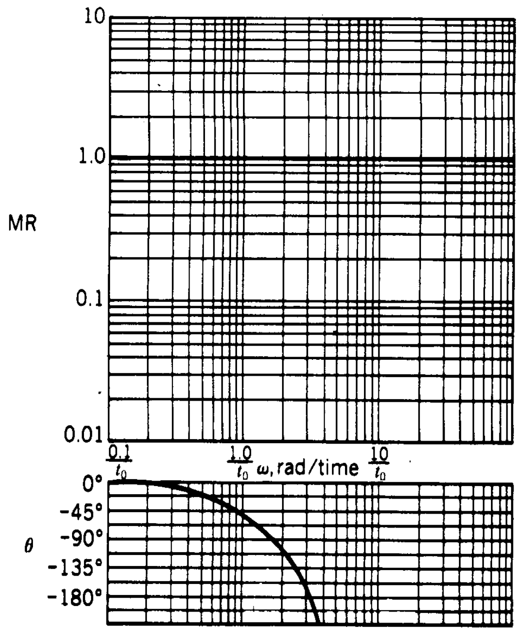
$$\boldsymbol{MR}_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \text{at} \quad \omega_{max} = \frac{\sqrt{1-2\xi^2}}{\tau}, \quad \xi \le \frac{1}{\sqrt{2}}$$

Step 3:
As $\omega \to 0$, $\theta \to 0$
As $\omega \nearrow \frac{1}{\tau}$, $\theta \nearrow -\frac{\pi}{2}$
As $\omega \searrow \frac{1}{\tau}$, $\theta \searrow -\frac{\pi}{2}$
As $\omega \to \infty$, $\theta \to -\pi$



Dead Time: $G_p(s) = e^{-t_0 s}$

$$\boldsymbol{AR} = 1, \quad \theta = -\omega t_0$$



Complex Systems: $G_p(s) = G_1(s) \cdots G_n(s)$

$$AR = |G_1(j\omega)| \cdots |G_n(j\omega)|$$

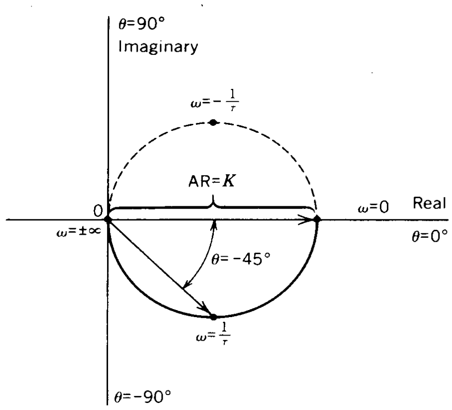
$$\theta = \angle G_1(j\omega) + \cdots + \angle G_n(j\omega)$$

10.3.2 Nyquist Plot

First order system: $G_p(s) = \frac{K}{\tau s+1}$

$$AR = \frac{K}{\sqrt{\omega^2\tau^2+1}}, \quad \theta = -\arctan(\omega\tau)$$

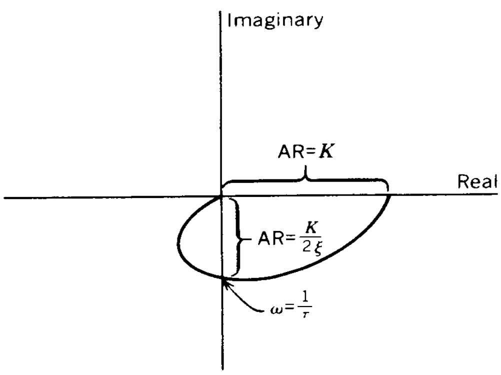
$\omega = 0$: $AR = K$ and $\theta = 0$
$\omega = \frac{1}{\tau}$: $AR = \frac{K}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$
$\omega = \infty$: $AR = 0$ and $\theta = -\frac{\pi}{2}$



Second order system: $G_p(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$

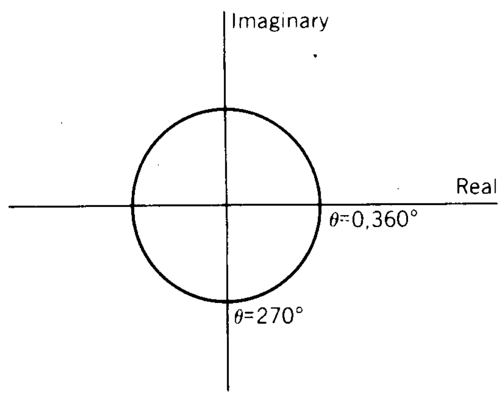
$$AR = \frac{K}{\sqrt{(1-\omega^2\tau^2)^2 + 4\xi^2\tau^2\omega^2}}, \quad \theta = -\arctan\left(\frac{2\xi\omega\tau}{1-\tau^2\omega^2}\right)$$

$\omega = 0$: $AR = K$ and $\theta = 0$
$\omega = \frac{1}{\tau}$: $AR = \frac{K}{2\xi}$ and $\theta = -\frac{\pi}{2}$
$\omega = \infty$: $AR = 0$ and $\theta = -\pi$



Dead Time: $G_p(s) = e^{-t_0 s}$

$$AR = 1, \quad \theta = -\omega t_0$$



Chapter 11

Stability of Dynamic Systems

Bounded Input Bounded Output Stability

Consider the I/O description of a linear time-invariant system:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau.$$

Def.: the system is bounded input bounded output (BIBO) stable if every bounded input results in a bounded output.

Theorem: A linear time invariant system is BIBO stable iff

$$\int_0^\infty |g(\tau)|d\tau < \infty.$$

Proof: ($\Leftarrow$) If u is bounded such that $|u(t)| \leq \boldsymbol{M}$ for all $t \geq 0$, then

$$|y(t)| = \left|\int_0^t g(t-\tau)u(\tau)d\tau\right| \leq \int_0^t |g(t-\tau)||u(\tau)|d\tau$$

$$\leq \boldsymbol{M}\int_0^\infty |g(\tau)|d\tau < \infty.$$

($\Rightarrow$) Suppose the contrary. Let

$$u(\tau) = \begin{cases} 1 & \text{if } g(t-\tau) \geq 0 \\ -1 & \text{if } g(t-\tau) < 0 \end{cases}.$$

Then

$$\lim_{t\to\infty} y(t) = \int_0^\infty |g(\tau)|d\tau = \infty.$$

This is a contradiction.

Let

$$G(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{a_0 s^n + \cdots + a_{n-1}s + a_n}.$$

Consider the step input $U(s) = \frac{1}{s}$ that is bounded. Then

$$Y(s) = \frac{N(s)}{a_0 s^n + \cdots + a_{n-1}s + a_n}\frac{1}{s} = \frac{\alpha_1}{s - s_1} + \cdots + \frac{\alpha_n}{s - s_n} + \frac{\beta}{s}$$

where s_i's are the poles of the system (Here we assumed $s_i \neq 0$ and all poles are distinct for simplicity).

$$\Downarrow$$

$$y(t) = \alpha_1 e^{s_1 t} + \cdots + \alpha_n e^{s_n t} + \beta.$$

Hence, the step response $y(t)$ is bounded if all s_i's are on the closed left half plane. However, for BIBO stability, s_i's are not allowed to be on the imaginary axes. To see this, consider $s_1 = 0$. Then

$$Y(s) = \frac{\alpha_1}{s} + \cdots + \frac{\alpha_n}{s - s_n} + \frac{\beta}{s^2}.$$

$$\Downarrow$$

$$y(t) = \alpha_1 + \cdots + \alpha_n e^{s_n t} + \beta t.$$

Since the poles determines the stability characteristics of the system, $a_0 s^n + \cdots + a_{n-1}s + a_n = 0$ is called characteristic polynomial (equation). Notice that by Cramer's rule,

$$G(s) = c^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}b + d = \frac{c^T adj(s\boldsymbol{I} - \boldsymbol{A})b + det(s\boldsymbol{I} - \boldsymbol{A})d}{det(s\boldsymbol{I} - \boldsymbol{A})}.$$

where $adj(s\boldsymbol{I} - \boldsymbol{A})$ is the adjoint of $s\boldsymbol{I} - \boldsymbol{A}$. Hence the characteristic polynomial is nothing more than the eigenvalue equation of $\boldsymbol{A}$.

General Stability Criterion: The system is BIBO stable iff all the poles have negative real parts.

Routh-Hurwitz Stability Criterion

Routh-Hurwitz stability criterion determines whether any roots of a polynomial equation:

$$a_0 s^n + \cdots + a_{n-1} s + a_n = 0$$

have positive real parts. In the following, we assume $a_0 > 0$ WLOG.

Routh array:

$$\begin{array}{rcccc} Row1 & a_0 & a_2 & a_4 & \cdots \\ 2 & a_1 & a_3 & a_5 & \cdots \\ 3 & b_0 & b_1 & b_2 & \cdots \\ 4 & c_0 & c_1 & \cdots & \cdots \\ 5 & d_0 & \cdots & & \end{array}$$

where

$$\begin{array}{lll} b_0 = \frac{a_1 a_2 - a_0 a_3}{a_1} & b_1 = \frac{a_1 a_4 - a_0 a_5}{a_1} & \cdots \\ c_0 = \frac{b_0 a_3 - a_1 b_1}{b_0} & c_1 = \frac{b_0 a_5 - a_1 b_2}{b_0} & \cdots \\ d_0 = \frac{c_0 b_1 - b_0 c_1}{c_0} & \cdots & \cdots \end{array}$$

Relationship between Routh array and the location of roots:

- If any element of the first column is negative, we have at least one root to the right of the imaginary axis.
- The number of sign changes in the elements of the first column is equal to the number of roots to the right of the imaginary axis.

Routh-Hurwitz Stability Criterion: The system is BIBO stable iff all the elements in the first column of the Routh array associated with the characteristic polynomial are positive.

Example: Consider the 2nd order characteristic polynomial equation:

$$a_0 s^2 + a_1 s + a_2 = 0$$

where $a_0 > 0$.

Routh array:

$$\begin{array}{rcc} Row1 & a_0 & a_2 \\ 2 & a_1 & \\ 3 & b_0 = a_2 & \end{array}$$

$$\Downarrow$$

For BIBO stability of the system, it must hold that $a_1, a_2 > 0$.

Chapter 12

Controllability and Observability

Controllability

Def.: z is said to be reachable from the origin if there is an input u that drives the state at the origin to z in $(0,t]$ for some t, i.e.

$$z = \int_0^t e^{A(t-\tau)} bu(\tau) d\tau.$$

Def.: A state space, or equivalently (A, b), is said to be controllable if each state is reachable.

For fixed $t > 0$, let $\Omega(t)$ be the set of all reachable state in $(0,t]$:

$$\Omega(t) = \left\{ x : x = \int_0^t e^{A(t-\tau)} bu(\tau) d\tau \right\}.$$

Notice that $\Omega(t)$ is a subspace. Let M_c be the set of all reachable state:

$$M_c = \cup_{t>0} \Omega(t).$$

Notice that M_c is a subspace and is called controllable subspace. Define the uncontrollable subspace as:

$$M_{uc} = M_c^{\perp} = (\cup_{t>0} \Omega(t))^{\perp} = \cap_{t>0} \Omega(t)^{\perp}.$$

Notice that $w \in \Omega(t)^{\perp}$ iff

$$0 = \left\langle w, \int_0^t e^{A(t-\tau)} bu(\tau) d\tau \right\rangle = \int_0^t \left\langle w, e^{A(t-\tau)} bu(\tau) \right\rangle d\tau$$

$$= \int_0^t \left\langle b^T e^{A^T(t-\tau)} w, u(\tau) \right\rangle d\tau.$$

Set

$$u(\tau) = b^T e^{A^T(t-\tau)} w, \quad 0 \le \tau \le t.$$

Then

$$\int_0^t \left\| b^T e^{A^T(t-\tau)} w \right\|^2 d\tau = 0$$

and thus

$$b^T e^{A^T(t-\tau)} w = 0, \quad 0 \le \tau \le t.$$

Clearly this is also sufficient for $w \in \Omega(t)^\perp$. Hence

$$\boldsymbol{M}_{uc} = \cap_{t>0} \left\{ w : b^T e^{A^T(t-\tau)} w = 0, \ 0 \le \tau \le t \right\}$$

$$= \left\{ w : 0 = b^T e^{A^T t} w = b^T \left[\sum_{i=1}^n a_i(t) (\boldsymbol{A}^T)^{i-1} \right] w = \sum_{i=1}^n a_i(t) b^T (\boldsymbol{A}^T)^{i-1} w, \ \forall t > 0 \right\}.$$

$$= \left\{ w : b^T (\boldsymbol{A}^T)^{i-1} w = 0, \ 1 \le i \le n \right\} = \left\{ w : \begin{bmatrix} b^T \\ b^T \boldsymbol{A}^T \\ \vdots \\ b^T (\boldsymbol{A}^T)^{n-1} \end{bmatrix} w = 0 \right\}$$

$$= \mathcal{N} \left(\begin{bmatrix} b^T \\ b^T \boldsymbol{A}^T \\ \vdots \\ b^T (\boldsymbol{A}^T)^{n-1} \end{bmatrix} \right).$$

Therefore since $\mathbf{R}^n = \mathcal{R}(\boldsymbol{T}^*) \oplus \mathcal{N}(\boldsymbol{T})$,

$$\boldsymbol{M}_c = \mathcal{R}(\boldsymbol{C})$$

where

$$\boldsymbol{C} = [b \ Ab \ \cdots \ A^{n-1} b]$$

that is called controllability matrix. To this end, we have the following theorem.

Theorem: TFAE

- $(\boldsymbol{A}, b)$ controllable

- $M_{uc} = \{0\}$
- $M_c = \mathbf{R}^n$
- $rank\, \mathcal{C} = n$

Ex: Let

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$\mathcal{C} = [b\ Ab] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore $det[b\ Ab] = 0$ and thus (A, b) is not controllable. Clearly

$$M_c = \mathcal{R}(\mathcal{C}) = span \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and thus

$$M_{uc} = M_c^{\perp} = span \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Observability

Def.: x_1 and x_2 are equivalent if, for every input u, the outputs associated with x_1 and x_2 are identical; i.e.

$$c^T e^{At} x_1 + \int_0^t c^T e^{A(t-\tau)} b u(\tau) d\tau + du(t) = c^T e^{At} x_2 + \int_0^t c^T e^{A(t-\tau)} b u(\tau) d\tau + du(t)$$

or

$$c^T e^{At} x_1 = c^T e^{At} x_2, \quad \forall t \geq 0.$$

Notice that two equivalent states are not distinguishable from their outputs.

Def.: A state space $\mathbf{R}^n$, or equivalently (c, A), is said to be observable if any two equivalent states are identical.

Notice that, if (c, A) is observable, any two states are distinguishable from their outputs.

Define the unobservable subspace as

$$M_{uo} = \{x \in \mathbf{R}^n : c^T e^{At} x = 0,\ \forall t \geq 0\}$$

which is the set of all states that are equivalent to 0. Notice that M_{uo} is a subspace, and x_1 and x_2 are equivalent iff $x_1 - x_2 \in M_{uo}$.

Define the observable subspace as

$$M_o = M_{uo}^{\perp}.$$

Suppose $x_1, x_2 \in M_o$ are equivalent. Then $x_1 - x_2 \in M_{uo}$ as well as $x_1 - x_2 \in M_o$. Hence $x_1 = x_2$.

Notice that

$$M_{uo} = \{x \in \mathbf{R}^n : c^T e^{At} x = 0, \ \forall t \geq 0\}$$

$$= \left\{ x : 0 = c^T e^{At} x = c^T \left[\sum_{i=1}^{n} \alpha_i(t) A^{i-1} \right] x = \sum_{i=1}^{n} \alpha_i(t) c^T A^{i-1} x, \ \forall t \geq 0 \right\}$$

$$= \left\{ x : c^T A^{i-1} x = 0, \ 1 \leq i \leq n \right\} = \left\{ x : \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{n-1} \end{bmatrix} x = 0 \right\} = \mathcal{N}(\mathcal{O})$$

where

$$\mathcal{O} = \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{n-1} \end{bmatrix}$$

that is called the observability matrix. Therefore since $\mathbf{R}^n = \mathcal{R}(T^*) \oplus \mathcal{N}(T)$,

$$M_o = \mathcal{R}\left([c \ A^T c \ \cdots \ (A^T)^{n-1} c] \right).$$

To this end, we have the following theorem.

Theorem: TFAE

- (c, A) observable
- $M_{uo} = \{0\}$
- $M_o = \mathbf{R}^n$
- $rank\ \mathcal{O} = n$

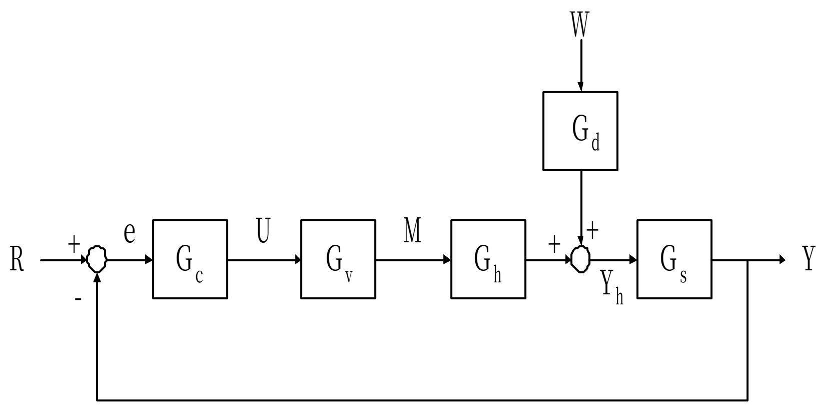
Notice that (c, A) is observable iff (A^T, c) is controllable.

Part III

Feedback Control Systems

Chapter 13

Feedback Control Loop



Elements in the feedback loop:

- Process:

$$Y_h(s) = G_h(s)M(s) + G_d(s)W(s)$$

- Measuring device:

$$Y(s) = G_s(s)Y_h(s)$$

- Controller:

$$U(s) = G_c(s)\epsilon(s)$$

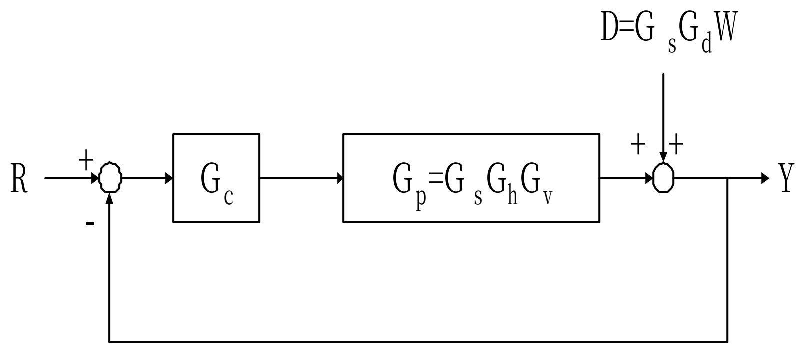
where

$$\epsilon(s) = \boldsymbol{R}(s) - \boldsymbol{Y}(s)$$

- Final Control Element:

$$\boldsymbol{M}(s) = \boldsymbol{G_v}(s)\boldsymbol{U}(s)$$

The above block diagram can be reduced to



Typical closed loop transfer functions:

- $\frac{Y(s)}{R(s)}$: the effect of reference input to the output
- $\frac{Y(s)}{D(s)}$: the effect of disturbance to the output

As far as the performance concerned, the goal of the control is to achieve

$$\frac{\boldsymbol{Y}(s)}{\boldsymbol{R}(s)} \approx 1 \quad \text{and} \quad \frac{\boldsymbol{Y}(s)}{\boldsymbol{D}(s)} \approx 0.$$

However, such achievement results in robustness problem.

From the block diagram:

$$\boldsymbol{Y}(s) = \boldsymbol{G_p}(s)\boldsymbol{G_c}(s)[\boldsymbol{R}(s) - \boldsymbol{Y}(s)] + \boldsymbol{D}(s)$$

$$\Downarrow$$

$$Y(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)} R(s) + \frac{1}{1 + G_p(s)G_c(s)} D(s)$$

$$\Downarrow$$

Complementary sensitivity:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)}$$

Sensitivity:

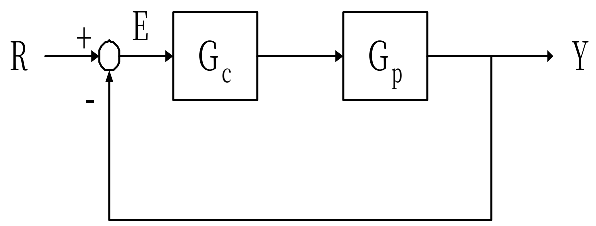
$$S(s) = \frac{Y(s)}{D(s)} = \frac{1}{1 + G_p(s)G_c(s)}$$

Notice that

$$S(s) + T(s) = 1.$$

Bode Stability Criterion:

Consider the feedback system:



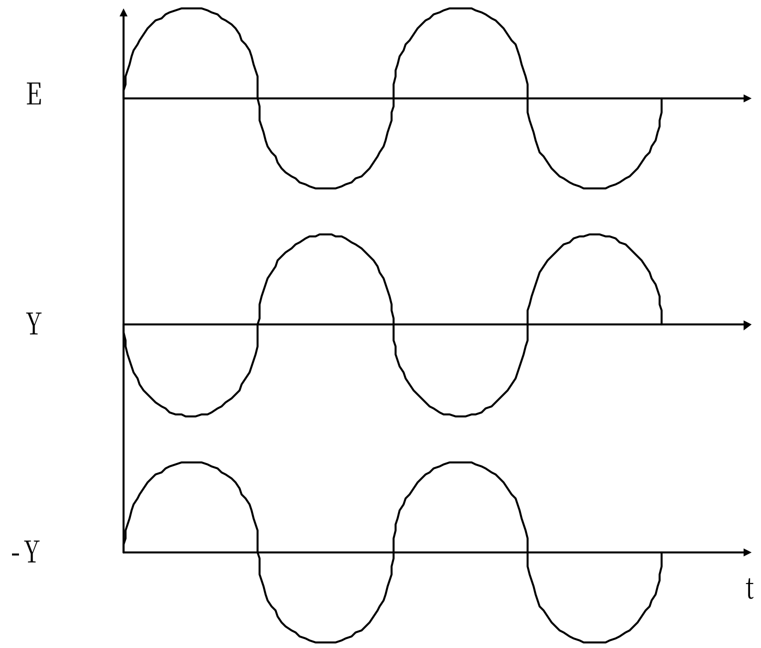
Open Loop Transfer Function (OLTF): $G_c G_p$

Critical (Crossover) Frequency, ω_c: frequency at which θ for OLTF, $G_c G_p$, is $-\pi$.

The closed loop system is stable if $AR(\omega_c) = |G_c(j\omega_c)G_p(j\omega_c)| < 1$. Otherwise, it is unstable.

Suppose $AR(\omega_c) = 1$.

- Disconnect feedback line and apply $E(t) = R(t) = R_0 \sin\omega_c t$. Then $Y(t) = Y_0 \sin(\omega_c t - \pi) = R_0 \sin(\omega_c t - \pi)$.
- Set $R(t) = 0$ and connect feedback line. Then $E(t) = -Y(t) = -R_0 \sin(\omega_c t - \pi) = R_0 \sin(\omega_c t)$ and, thus, $Y(t) = R_0 \sin(\omega_c t - \pi)$.



Clearly, after setting $R(t) = 0$ and connecting feedback line, the magnitude of oscillation will decay (grow) if $AR(\omega_c) \underset{(>)}{<} 1$.

<u>Nyquist Stability Criterion:</u>

If N is the number of times that Nyquist plot encircles $(-1, 0)$ in the clockwise direction and P is the number of unstable OLTF poles, then $Z = N + P$ is the number of unstable CLTF poles (N may be negative if Nyquist plot encircles $(-1, 0)$ in the counter-clockwise direction).

Chapter 14

PID Control

14.1 PID controllers

Consider the plant

$$G_p(s) = \frac{N_p(s)}{D_p(s)}$$

where $N_p(s)$ and $D_p(s)$ are numerator and denominator polynomials.

P controller

Proportional (P) controller: $u = K_c e = K_c(r - y) \Rightarrow G_c(s) = K_c$.

Suppose $R(s) = 0$ and the closed loop system is stable. Consider the step disturbance $D(s) = \frac{1}{s}$. Then

$$Y(s) = \frac{1}{1 + K_c G_p(s)} D(s) = \frac{D_p(s)}{D_p(s) + K_c N_p(s)} \frac{1}{s} = \frac{\alpha_1}{s - s_1} + \cdots + \frac{\alpha_n}{s - s_n} + \frac{\beta}{s}$$

where s_i's are the poles of the closed loop system (Here all the closed loop poles are distinct for simplicity).

$$\Downarrow$$

$$y(t) = \alpha_1 e^{s_1 t} + \cdots + \alpha_n e^{s_n t} + \beta.$$

Since the closed loop system is stable,

$$\lim_{t \to \infty} y(t) = \beta.$$

β is called the steady state offset. Hence for step disturbance, the proportional controller alone cannot remove the effects of the disturbance to the output.

PI controller

Proportional-Integral (PI) controller: $u = K_c e + \frac{K_c}{\tau_I}\int_{\tau=0}^{t} e(\tau)d\tau \Rightarrow G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$.

Suppose $R(s) = 0$ and the closed loop system is stable. Consider the step disturbance $D(s) = \frac{1}{s}$. Then

$$Y(s) = \frac{1}{1 + K_c\left(1 + \frac{1}{\tau_I s}\right)G_p(s)}D(s) = \frac{\tau_I s D_p(s)}{\tau_I s D_p(s) + K_c(1 + \tau_I s)N_p(s)}\frac{1}{s}$$

$$= \frac{\alpha_1}{s - s_1} + \cdots + \frac{\alpha_n}{s - s_n}$$

where s_i's are the poles of the closed loop system (Here all the closed loop poles are distinct for simplicity).

$$\Downarrow$$

$$y(t) = \alpha_1 e^{s_1 t} + \cdots + \alpha_n e^{s_n t}.$$

Since the closed loop system is stable,

$$\lim_{t\to\infty} y(t) = 0.$$

The steady state offset can be eliminated using the integral action.

PID controller

Proportional-Integral-Derivative (PID) controller: $u = K_c e + \frac{K_c}{\tau_I}\int_{\tau=0}^{t} e(\tau)d\tau + K_c\tau_D\frac{de}{dt} \Rightarrow G_c(s) = K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)$.

The derivative mode is used to anticipate the future behavior of the error $e(t)$ from its derivative mode. However, due to noise, it is impossible to compute the meaningful derivative value. Hence the following approximation is often used.

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)\left(\frac{\tau_D s + 1}{\alpha\tau_D s + 1}\right)$$

where $\alpha \in [0.05, 0.1]$ typically.

14.2 Closed Loop Systems Stability with PID Controllers

Routh-Hurwitz Criterion

Example 1:

- 1st order plant: $G_p(s) = \frac{K}{\tau s+1}$
- PI controller: $G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right) = \frac{K_c \tau_I s + K_c}{\tau_I s}$

Characteristic equation:

$$\tau\tau_I s^2 + (1 + K_c K)\tau_I s + K_c K = 0$$

From Routh-Hurwitz criterion, the coefficients must be all positive. Hence, it must hold that $K_c, \tau_I > 0$.

Example 2:

- 3rd order plant: $G_p(s) = \frac{1}{(s+1)^3}$
- P controller: $G_c(s) = K_c$

$$Y(s) = \frac{G_p(s)G_c(s)}{1 + G_p(s)G_c(s)}R(s) + \frac{1}{1 + G_p(s)G_c(s)}D(s)$$

$$Y(s) = \frac{\frac{K_c}{(s+1)^3}}{1 + \frac{K_c}{(s+1)^3}}R(s) + \frac{1}{1 + \frac{K_c}{(s+1)^3}}D(s)$$

$$\Downarrow$$

Characteristic equation:

$$s^3 + 3s^2 + 3s + 1 + K_c = 0.$$

Routh array:

Row1	1	3
2	3	$1 + K_c$
3	$\frac{8-K_c}{3}$	
4	$1 + K_c$	

$$\Downarrow$$

For BIBO stability of the closed loop system, it must hold that

$$\frac{8 - K_c}{3} > 0, \quad 1 + K_c > 0.$$

$$\Downarrow$$

$$-1 < K_c < 8.$$

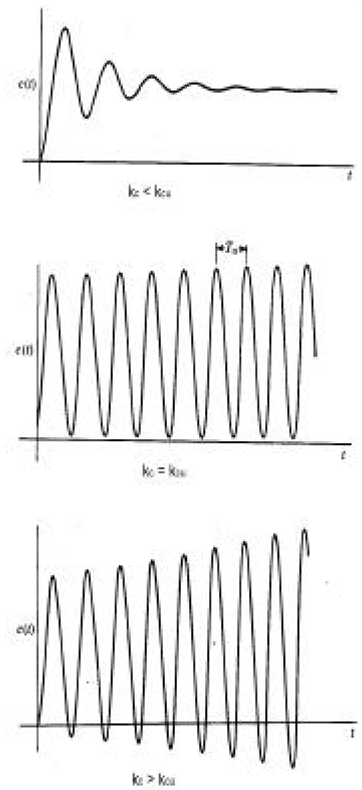
Remark: Similar bounds can also be obtained from the Bode and Nyquist stability criterions.

14.3 Tuning Based on Ultimate Gains: Ziegler-Nichols (Quarter Decay Ratio) Method

From Example 2 in the previous section: When P-only control is applied, the stabilizing K_c values are "usually" upper bounded by a positive constant. (There exist cases where such upper bound doesn't exist)

The upper bound is called the ultimate gain denoted K_{cu}.

When $K_c = K_{cu}$, the closed loop has some of its poles on the imaginary axis. In that case, the closed loop system is called marginally stable.



Example (Example 2 in the previous section is continued): If $K_c = 8$, characteristic equation becomes

$$s^3 + 3s^2 + 3s + 9 = s^2(s+3) + 3(s+3) = (s^2+3)(s+3) = 0.$$

$\Rightarrow$ Poles: $-3, \pm j\sqrt{3}$.

Computation of Ultimate Gain Using Mathematical Model

Let $L(K_c, s)$ be the characteristic polynomial.

$$L(K_c, j\omega) = 0$$
$$\Downarrow$$
$$ReL(K_c, j\omega) = 0 \qquad ImL(K_c, j\omega) = 0$$
$$\Downarrow$$
$$K_c = K_{cu}, \qquad \omega = \omega_u.$$

This computation gives the ultimate gain K_{cu} as well as the poles $p = \pm j\omega_u$ on the imaginary axis at $K_c = K_{cu}$.

ω_u is called ultimate frequency.

$T_u = \frac{2\pi}{\omega_u}$ is called ultimate period.

Example 1 (Example 2 in the previous section is continued):

$$L(K_c, j\omega) = -j\omega^3 - 3\omega^2 + 3j\omega + 1 + K_c = (1 + K_c - 3\omega^2) + j(3\omega - \omega^3) = 0$$
$$\Downarrow$$
$$1 + K_c - 3\omega^2 = 0, \qquad \omega(3 - \omega^2) = 0$$
$$\Downarrow$$
$$\omega_u = \pm\sqrt{3}, 0, \qquad K_{cu} = 8, -1.$$

Example 2:

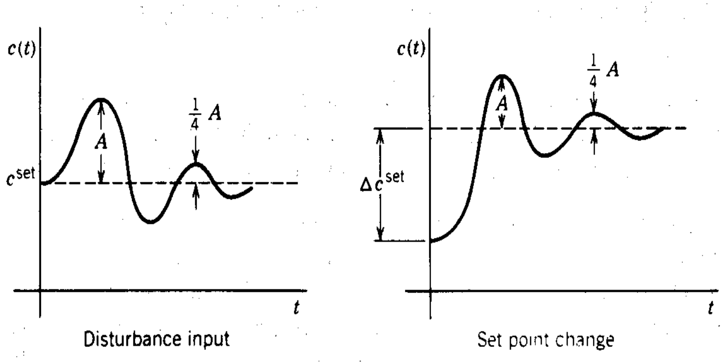
- 1st order plant with dead time: $G_p(s) = \frac{2e^{-s}}{s+1}$
- P controller: $G_c(s) = K_c$

$$\Downarrow$$
$$L(K_c, s) = s + 1 + 2K_c e^{-s}$$
$$\Downarrow$$
$$j\omega+1+2K_c e^{-j\omega} = j\omega+1+2K_c(\cos\omega - j\sin\omega) = (1+2K_c\cos\omega)+j(\omega-2K_c\sin\omega) = 0$$
$$\Downarrow$$
$$1 + 2K_c\cos\omega = 0, \quad \omega - 2K_c\sin\omega = 0$$
$$\Downarrow$$
$$\omega_u = 2.0288, \quad K_{cu} = 1.1309.$$

Evaluation of Ultimate Gain by Experiment

1. Switch off the integral and derivative actions so as to have a P controller.
2. Increase P gain until the loop oscillates with constant amplitude. Apply a small set point change when it is hard to observe the process response.
3. When the sustained oscillation is achieved, the corresponding gain is K_{cu} and the period of the oscillation is T_u.

Based on K_{cu} and T_u, Ziegler-Nichols proposed to tune the PID parameters so that the ratio of the amplitudes of two successive oscillations is $\frac{1}{4}$.



Controller Type	K_c	τ_I	τ_D
P	$\frac{K_{cu}}{2}$		
PI	$\frac{K_{cu}}{2.2}$	$\frac{T_u}{1.2}$	
PID	$\frac{K_{cu}}{1.7}$	$\frac{T_u}{2}$	$\frac{T_u}{8}$

14.4 Tuning Based on First-Order Plus Dead-Time (FOPDT) Model

Many chemical processes can be approximated by FOPDT model.

$$Y(s) = \frac{Ke^{-ds}}{\tau s + 1} U(s)$$

where K, d, τ are parameters.

14.4.1 FOPDT Model from Step Response of Plant

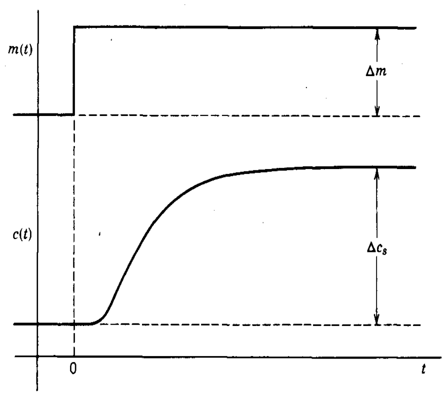
Response of FOPDT model to the step with magnitude Δm:

$$y(t) = \Delta m K (1 - e^{-\frac{t-d}{\tau}}) u(t - d)$$

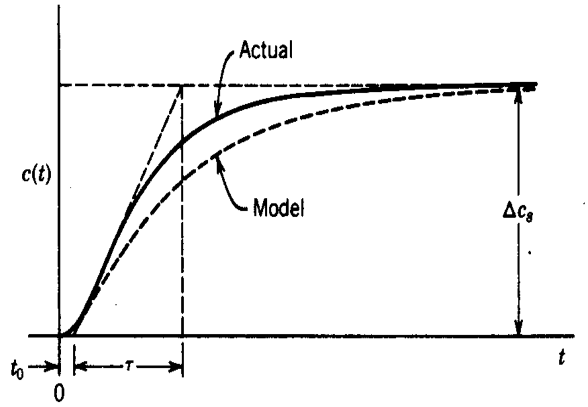
where $u(t)$ is the unit step function.

Identification of FOPDT model parameters through step testing

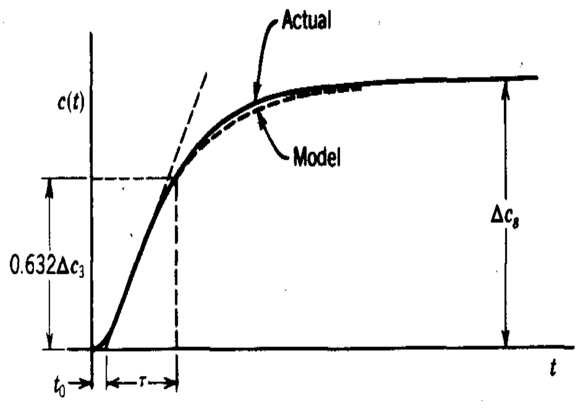
- Step 1: Record the step response of plant:



- Step 2: Evaluate the steady state gain $K = \frac{\Delta y_s}{\Delta m}$.
- Step 3
  - Method 1: Draw tangent line at the point of maximum rate of change. Then determine d, τ as in the figure.

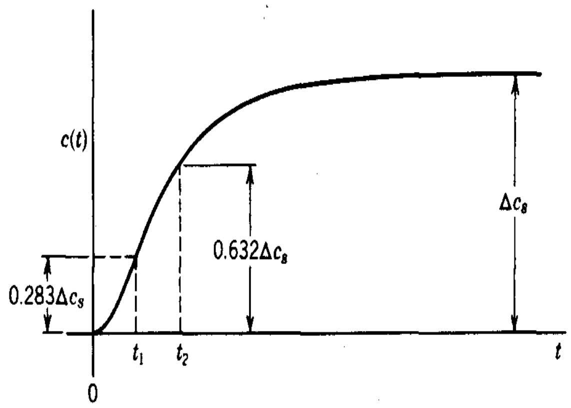


  - Method 2: Draw tangent line at the point of maximum rate of change. Then determine d, τ as in the figure.



– Method 3: Find t_1, t_2 as in the figure. Then

$$\tau = 1.5(t_2 - t_1), \quad d = t_2 - \tau$$



14.4.2 Ziegler-Nichols (Quarter Decay Ratio) Method Based on FOPDT Model

Ziegler and Nichols also proposed quarter-decay ratio tuning rule based on FOPDT model:

Controller Type	K_c	τ_I	τ_D
P	$\frac{1}{K}\left(\frac{d}{\tau}\right)^{-1}$		
PI	$\frac{0.9}{K}\left(\frac{d}{\tau}\right)^{-1}$	$3.33d$	
PID	$\frac{1.2}{K}\left(\frac{d}{\tau}\right)^{-1}$	$2.0d$	$0.5d$

14.4.3 Cohen-Coon Method Based on FOPDT Model

Cohen and Coon considered the quarter decay ratio, small offset as well as small integral error, and proposed the following empirical tuning rules:

Controller Type	K_c	τ_I	τ_D
P	$\frac{1}{K}\frac{\tau}{d}\left(1+3\frac{d}{\tau}\right)$		
PI	$\frac{1}{K}\frac{\tau}{d}\left(0.9+\frac{d}{12\tau}\right)$	$\frac{d\left(30+3\frac{d}{\tau}\right)}{9+20\frac{d}{\tau}}$	
PID	$\frac{1}{K}\frac{\tau}{d}\left(\frac{16\tau+3d}{12\tau}\right)$	$\frac{d\left(32+6\frac{d}{\tau}\right)}{13+8\frac{d}{\tau}}$	$\frac{4d}{11+2\frac{d}{\tau}}$

14.5 Tuning Based on Frequency Response

Connection between ultimate gain and critical frequency:

$$1+K_cG_p(j\omega)=0 \quad \Rightarrow \quad K_cG_p(j\omega)=-1 \quad \Rightarrow \quad K_{cu},\omega_u.$$

On the other hand,

$$\angle K_cG_p(j\omega_c)=\angle G_p(j\omega_c)=-\pi, \quad \forall K_c>0.$$

Pick $K_c^*=\frac{1}{|G_p(j\omega_c)|} \Rightarrow |K_c^*G_p(j\omega_c)|=1 \Rightarrow K_c^*G_p(j\omega_c)=-1 \Rightarrow K_c^*=K_{cu}$ and $\omega_c=\omega_u$.

$$\Downarrow$$

Ziegler-Nichols method can be used for PID controller tuning.

Notice that

$$K_c<K_{cu} \quad \Rightarrow \quad |K_cG_p(j\omega_c)|<1 \quad \Rightarrow \quad \text{stable}$$

$$K_c>K_{cu} \quad \Rightarrow \quad |K_cG_p(j\omega_c)|>1 \quad \Rightarrow \quad \text{unstable.}$$

Controller gain tuning based on frequency response

- Gain margin (GM): the ratio between K_c and K_{cu}

$$GM=\frac{K_{cu}}{K_c}$$

  Usually K_c is chosen such that $GM>1.5$.

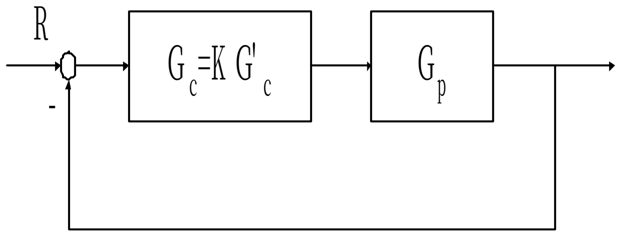
- Phase margin (PM): the difference between $-\pi$ and phase angle at the frequency where AR is 1.

$$PM=\pi+\theta|_{AR=1}$$

  Usually K_c is chosen such that $PM>\frac{\pi}{4}$.

14.6 Root Locus Techniques

Consider the feedback system:



Open Loop Transfer Function (OLTF): $KG'_cG_p = K\frac{N}{D}$
Closed Loop Transfer Function (CLTF): $\frac{KG'_cG_p}{1+KG'_cG_p} = \frac{KN}{D+KN}$

- Zeros of CLTF = zeros of OLTF
- Poles of CLTF = $\begin{cases} \text{poles of OLTF} & \text{if } K = 0 \\ \text{zeros of OLTF} & \text{if } K \to \infty \\ ? & \text{otherwise} \end{cases}$

Question: How do the poles of CLTF change as K increase from 0?
Answer: Root Locus.
Example:

- 2nd order plant: $G_p(s) = \frac{1}{(3s+1)(s+1)}$
- P controller: $K = K_c$ and $G'_c(s) = 1$

$$\Downarrow$$

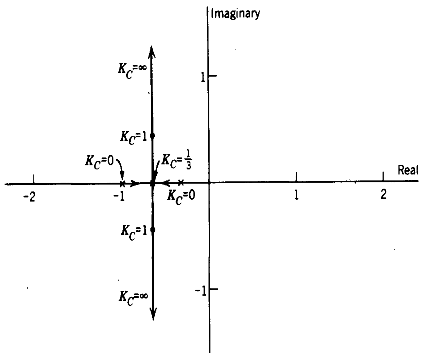
Characteristic Polynomial:

$$(3s+1)(s+1) + K_c = 0$$

$$\Downarrow$$

$$s_{1,2} = -\frac{2}{3} \pm \frac{1}{3}\sqrt{1 - 3K_c}.$$

$K_c = 0$	$s_{1,2} = -1, -\frac{1}{3}$
$0 < K_c \leq \frac{1}{3}$	$s_{1,2} =$ real number
$K_c = \frac{1}{3}$	$s_{1,2} = -\frac{2}{3}$
$K_c > \frac{1}{3}$	$s_{1,2} = -\frac{2}{3} \pm j\alpha$



14.6.1 Rules for plotting root locus

Suppose

$$KG_cG_p = K'\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

where z_i, p_i's are zeros and poles of OLFT, respectively.

$$\Downarrow$$

Characteristic equation:

$$0 = (s-p_1)\cdots(s-p_n) + K'(s-z_1)\cdots(s-z_m).$$

- As $K' \to 0$, characteristic equation:

  $$0 = (s-p_1)\cdots(s-p_n) + K'(s-z_1)\cdots(s-z_m) \approx (s-p_1)\cdots(s-p_n)$$

  CLTF poles = OLTF poles

- As $K' \to \infty$, characteristic equation:

  $$0 = \frac{1}{K'}(s-p_1)\cdots(s-p_n) + (s-z_1)\cdots(s-z_m) \approx (s-z_1)\cdots(s-z_m)$$

  CLTF poles = OLTF zeros

$$\Downarrow$$

CLTF poles start at the OLTF poles and terminate at the OLTF zeros.

Two governing conditions

Characteristic equation can be written as

$$-1 = K' \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}.$$

Before we proceed, notice that

$$w_1 w_2 = |w_1| e^{j\phi_1} |w_2| e^{j\phi_2} = |w_1||w_2| e^{j(\phi_1 + \phi_2)}.$$

1. Magnitude condition:

$$1 = K' \frac{|s - z_1| \cdots |s - z_m|}{|s - p_1| \cdots |s - p_n|}$$

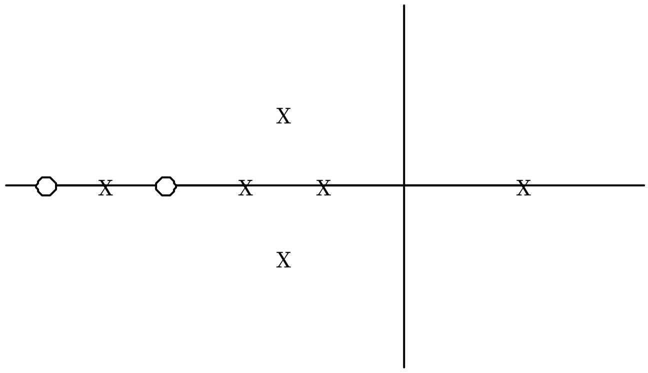
$$\Downarrow$$

$$K' = \frac{|s - p_1| \cdots |s - p_n|}{|s - z_1| \cdots |s - z_m|}.$$

2. Angle condition:

$$\angle \left[K' \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)} \right]$$
$$= \angle(s - z_1) + \cdots + \angle(s - z_m) - \angle(s - p_1) - \cdots - \angle(s - p_n)$$
$$= \angle - 1 = -\pi + 2k\pi.$$

- Rule #1: The number of loci is equal to the number of OLTF poles.
- Rule #2: The root loci begin at the OLTF poles and terminate at the OLTF zeros. Since $n - m$ OLTF zeros are at infinity, $n - m$ loci diverges.
- Rule #3 (Loci on real axis): A point on the real axis is a part of a root locus when sum of the number of poles and zeros to the right of the point on the real axis is odd.

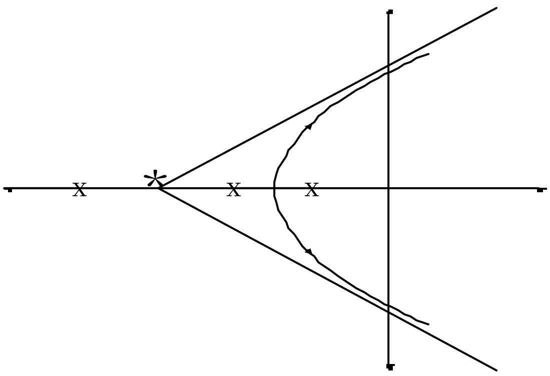


- Rule #4 (Asymptotes (zeros at infinity): $n-m$ diverging loci approach to infinity along the straight lines, called asymptotes, that pass through the center of gravity

$$\gamma = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

with angles:

$$\frac{\pi(2k+1)}{n-m}, \quad k = 0, 1, \cdots, n-m-1.$$



Proof: For large s,

$$-K' = \frac{(s-p_1)\cdots(s-p_n)}{(s-z_1)\cdots(s-z_m)} = s^{n-m} \frac{\left(1-p_1\frac{1}{s}\right)\cdots\left(1-p_n\frac{1}{s}\right)}{\left(1-z_1\frac{1}{s}\right)\cdots\left(1-z_m\frac{1}{s}\right)}$$

$$\approx s^{n-m}\frac{\left(1-(p_1+\cdots+p_n)\frac{1}{s}\right)}{\left(1-(z_1+\cdots+z_m)\frac{1}{s}\right)}$$
$$\approx s^{n-m}\left(1-(p_1+\cdots+p_n)\frac{1}{s}\right)\left(1+(z_1+\cdots+z_m)\frac{1}{s}\right)$$
$$\approx s^{n-m}-(p_1+\cdots+p_n-z_1-\cdots-z_m)s^{n-m-1}\approx(s-\gamma)^{n-m}.$$

Angle condition:

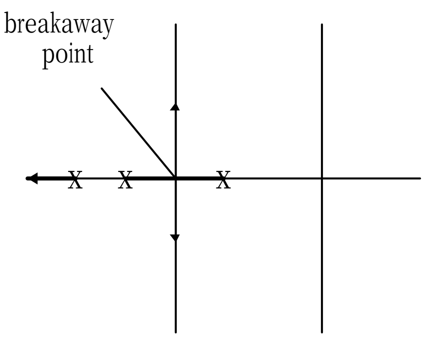
$$-K' = (s-\gamma)^{n-m} \quad \Rightarrow \quad \angle(s-\gamma)=\frac{\pi(2k+1)}{n-m}.$$

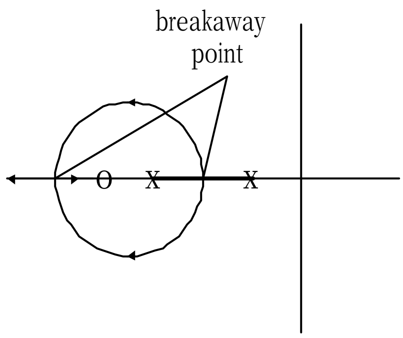
- Rule #5 (Breakaway point): The points on the real axis where loci meet and leave, or enter from the complex region are called breakaway points. The breakaway points are determined by

$$0=\frac{d}{ds}\left(\frac{1}{G_c(s)G_p(s)}\right)$$

or

$$\sum_{i=1}^{m}\frac{1}{s-z_i}=\sum_{j=1}^{n}\frac{1}{s-p_j}$$



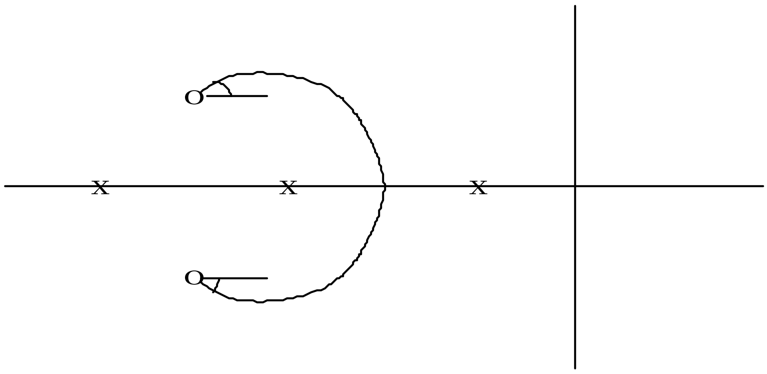


- Rule #6 (Angles of departure and arrival): When a locus leaves from a complex conjugate pole p_k, the angle of departure is

$$\pi + \sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(p_k - p_j).$$

When a locus arrives at a complex conjugate zero z_k, the angle of arrival is

$$-\pi + \sum_{j=1}^{n} \angle(z_k - p_j) - \sum_{\substack{i=1 \\ i \neq k}}^{m} \angle(z_k - z_i).$$



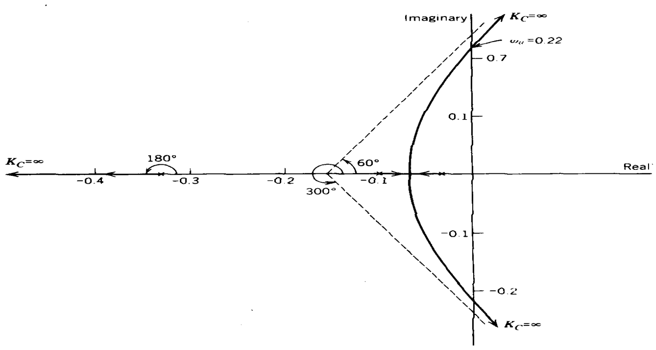
Example 1:

- 3rd order plant: $G_p(s) = \frac{0.8}{(3s+1)(30s+1)(10s+1)}$
- PI controller: $K = K_c$ and $G_c(s) = \left(1 + \frac{1}{40s}\right)$

$$\Downarrow$$

$$\text{OLTF} = \frac{\left(\frac{0.8}{900}\right)K_c\left(s+\frac{1}{40}\right)}{s\left(s+\frac{1}{3}\right)\left(s+\frac{1}{30}\right)\left(s+\frac{1}{10}\right)}$$

OLTF poles: $0, -\frac{1}{3}, -\frac{1}{30}, -\frac{1}{10}$
OLTF zero: $-\frac{1}{40}$



Breakaway point

$$\frac{1}{s}+\frac{1}{s+\frac{1}{3}}+\frac{1}{s+\frac{1}{30}}+\frac{1}{s+\frac{1}{10}}=\frac{1}{s+\frac{1}{40}}$$

$\Rightarrow -0.058.$

Asymptotes:

$$\gamma=\frac{0-\frac{1}{3}-\frac{1}{30}-\frac{1}{10}+\frac{1}{40}}{3}=-0.3587.$$

Angles: $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Example 2:

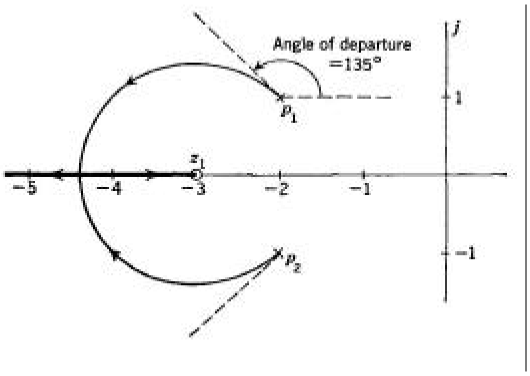
- 2rd order plant: $G_p(s)=\frac{1}{s^2+4s+5}$
- PD controller: $K=K_c$ and $G_c(s)=\left(1+\frac{1}{3}s\right)$

$$\Downarrow$$

OLTF $=\frac{K_c\left(1+\frac{1}{3}s\right)}{(s^2+4s+5)}=\frac{1}{3}K_c\frac{(s+3)}{(s+2+j)(s+2-j)}$

OLTF poles: $-2\pm j$

OLTF zero: -3



Breakaway point

$$\frac{d}{ds}\left[-\frac{s^2+4s+5}{(s+3)}\right]=0$$

$\Rightarrow\ -4.41(\boldsymbol{O}), -1.59(\boldsymbol{X})$.

Angle of departure

$p=-2+j$:

$$\pi+\angle(-2+j+3)-\angle(-2+j+2+j)=\pi+\angle(1+j)-\angle(2j)=\pi+\frac{\pi}{4}-\frac{\pi}{2}=\frac{3}{4}\pi.$$

$p=-2-j$:

$$\pi+\angle(-2-j+3)-\angle(-2-j+2-j)=\pi+\angle(1-j)-\angle(-2j)=\pi-\frac{\pi}{4}+\frac{\pi}{2}=\frac{5}{4}\pi.$$

Chapter 15

State Feedback Control Systems

15.1 Pole-Zero Cancellation

Consider the unstable plant:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{s-1}.$$

If we connect the plant serially with the controller

$$G_c = \frac{U(s)}{V(s)} = \frac{s-1}{s+1} = 1 - \frac{2}{s+1},$$

the resulting system

$$G(s) = \frac{1}{s-1}\frac{s-1}{s+1} = \frac{1}{s+1}$$

is stable. However this design doesn't work. To see this, consider the controller and plant state equations:

$$\dot{x}_1 = -x_1 - 2v$$

$$u = x_1 + v$$

$$\dot{x}_2 = x_2 + u = x_2 + x_1 + v$$

$$y = x_2.$$

Notice that

$$x_1(t) = e^{-t}x_{10} - 2e^{-t} * v.$$

Taking LT of the equations,

$$Y(s) = X_2(s) = \frac{x_{20}}{s-1} + \frac{x_{10}}{(s-1)(s+1)} + \frac{V(s)}{s+1},$$

and thus

$$y = x_2 = e^t x_{20} + \frac{1}{2}(e^t - e^{-t})x_{10} + e^{-t} * v.$$

Notice that in the input output model where the initial condition is assumed to be zero, the output is bounded. However it is difficult to keep the initial condition at zero every time and the above control will not work. Indeed in this case, there is a direct pole-zero cancellation and the behavior of unstable state is hidden in the input-output (external) behavior. Hence to design satisfactory control system, one need to keep track of internal (all the states') behavior. If all states are stable, such a system is called internally stable.

15.2 Controller Canonical Form

Theorem: Suppose $(\boldsymbol{A}, b)$ is controllable. Let

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_1\boldsymbol{A} \\ \vdots \\ \boldsymbol{P}_1\boldsymbol{A}^{n-1} \end{bmatrix}$$

where

$$\boldsymbol{P}_1 = [0 \ \cdots \ 0 \ 1][b \ \boldsymbol{A}b \ \cdots \ \boldsymbol{A}^{n-1}b]^{-1}.$$

Then the transformation $z = \boldsymbol{P}x$ leads to the controller canonical form

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

where

$$\chi_A(s) = det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0.$$

Proof: Notice that

$$z_1 = \boldsymbol{P}_1 x$$

and thus

$$\dot{z}_1 = \boldsymbol{P}_1 \dot{x} = \boldsymbol{P}_1 \boldsymbol{A} x + \boldsymbol{P}_1 b u = \boldsymbol{P}_1 \boldsymbol{A} x + [0 \ \cdots \ 0 \ 1] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u = \boldsymbol{P}_1 \boldsymbol{A} x = z_2.$$

Therefore

$$\dot{z}_2 = \boldsymbol{P}_1 \boldsymbol{A} \dot{x} = \boldsymbol{P}_1 \boldsymbol{A}^2 x + \boldsymbol{P}_1 \boldsymbol{A} b u = \boldsymbol{P}_1 \boldsymbol{A}^2 x + [0 \ \cdots \ 0 \ 1] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u = \boldsymbol{P}_1 \boldsymbol{A}^2 x = z_3.$$

Continuing this process, we obtain

$$\dot{z}_{n-1} = \boldsymbol{P}_1 \boldsymbol{A}^{n-2} \dot{x} = \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + \boldsymbol{P}_1 \boldsymbol{A}^{n-2} b u = \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + [0 \ \cdots \ 0 \ 1] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$= \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x = z_n.$$

Moreover by Cayley-Hamilton theorem,

$$\dot{z}_n = \boldsymbol{P}_1 \boldsymbol{A}^{n-1} \dot{x} = \boldsymbol{P}_1 \boldsymbol{A}^n x + \boldsymbol{P}_1 \boldsymbol{A}^{n-1} b u$$

$$= \boldsymbol{P}_1(-a_n - a_{n-1}\boldsymbol{A} - \cdots - a_1 \boldsymbol{A}^{n-1}) x + [0 \ \cdots \ 0 \ 1] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$= -a_n \boldsymbol{P}_1 x - a_{n-1} \boldsymbol{P}_1 \boldsymbol{A} x - \cdots - a_1 \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + u = -a_n z_1 - a_{n-1} z_2 - \cdots - a_1 z_n + u.$$

15.3 Pole Placement

If $(\boldsymbol{A}, b)$ is controllable, the n poles of the feedback system can be located in any places of the complex plane through static state feedback. Such placement of poles in any desired locations is called pole placement. In this section we consider internal stabilization of the system through the pole placement.

Consider the static state feedback control law:

$$u = -k^T x.$$

Then the closed system becomes

$$\dot{x} = \boldsymbol{A}x - bk^T x = (\boldsymbol{A} - bk^T)x$$

where $k = [k_n \; k_{n-1} \; \cdots \; k_1]^T$.

In this section we present two different methods of pole placement. For this let $\{p_i\}_{i=1}^n$ be the set of desired poles. Define

$$0 = (s - p_1)(s - p_2) \cdots (s - p_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n = \chi_\alpha(s).$$

<u>Bass-Gura Formula:</u>

A way to achieve the pole placement is to first transform the system representation into controller form as shown in the previous section. Then the feedback system is

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n - k_n^c & -a_{n-1} - k_{n-1}^c & -a_{n-2} - k_{n-2}^c & \cdots & -a_1 - k_1^c \end{bmatrix} z$$

and the characteristic equation is

$$s^n + (a_1 + k_1^c)s^{n-1} + (a_2 + k_2^c)s^{n-2} + (a_3 + k_3^c)s^{n-3} + \cdots + (a_n + k_n^c) = 0.$$

Hence it is clear that the controller gain must be

$$k^c = \alpha - a$$

where $\alpha = [\alpha_n \ \alpha_{n-1} \ \cdots \ \alpha_1]^T$ and $a = [a_n \ a_{n-1} \ \cdots \ a_1]^T$. Notice that

$$u = -(k^c)^T z = -(k^c)^T \boldsymbol{P} x.$$

Hence the feedback gain must be

$$k^T = (\alpha - a)^T \boldsymbol{P}.$$

This formula is called the Bass-Gura formula.

Ackermann's Formula

Notice that by Cayley-Hamilton theorem,

$$\boldsymbol{P}_1 \chi_\alpha(\boldsymbol{A}) x = \boldsymbol{P}_1 \boldsymbol{A}^n x + \alpha_1 \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + \alpha_2 \boldsymbol{P}_1 \boldsymbol{A}^{n-2} x + \cdots + \alpha_n \boldsymbol{P}_1 x$$

$$= -a_1 \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x - a_2 \boldsymbol{P}_1 \boldsymbol{A}^{n-2} x - \cdots - a_n \boldsymbol{P}_1 x + \alpha_1 \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + \alpha_2 \boldsymbol{P}_1 \boldsymbol{A}^{n-2} x + \cdots + \alpha_n \boldsymbol{P}_1 x$$

$$= (\alpha_1 - a_1) \boldsymbol{P}_1 \boldsymbol{A}^{n-1} x + (\alpha_2 - a_2) \boldsymbol{P}_1 \boldsymbol{A}^{n-2} x + \cdots + (\alpha_n - a_n) \boldsymbol{P}_1 x$$

$$= k_1^c z_n + k_2^c z_{n-1} + \cdots + k_n^c z_1 = (k^c)^T z = k^T x.$$

Hence,

$$k^T = \boldsymbol{P}_1 \chi_\alpha(\boldsymbol{A}).$$

This formula is called the Ackermann's formula.

Ex: Consider the BIBO unstable system described by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Notice that

$$\chi_A(s) = s^2.$$

Suppose we want to locate the closed loop system poles at -2. Then

$$\chi_\alpha(s) = s^2 + 4s + 4.$$

Moreover

$$[b \ \boldsymbol{A}b] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [b \ \boldsymbol{A}b]^{-1}$$

and thus

$$\boldsymbol{P}_1 = [0 \ 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [1 \ 0].$$

Using Bass-Gura formula,

$$k^T = [4\ 4] \begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_1 \boldsymbol{A} \end{bmatrix} = [4\ 4] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [4\ 4].$$

Using Ackermann's formula,

$$k^T = \boldsymbol{P}_1 \chi_{\alpha}(\boldsymbol{A}) = [1\ 0] \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 + 4 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= [1\ 0] \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} = [4\ 4].$$

Chapter 16

Observer and Output Feedback

Asymptotic Observer (State Estimator)

Goal: Based on the input-output data, find the state estimate that converges to the actual state.

Asymptotic Observer (State Estimator):

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + bu(t) + l(\tilde{y}(t) - y(t))$$

where the predicted output $\tilde{y}(t) = c^T\tilde{x}(t)$.

$$\Downarrow$$

$$\dot{\tilde{x}}(t) = (A + lc^T)\tilde{x}(t) + bu(t) - ly(t).$$

Notice that the observer gives the state estimate $\tilde{x}$ from the I/O pair (u, y).

Define state estimation error

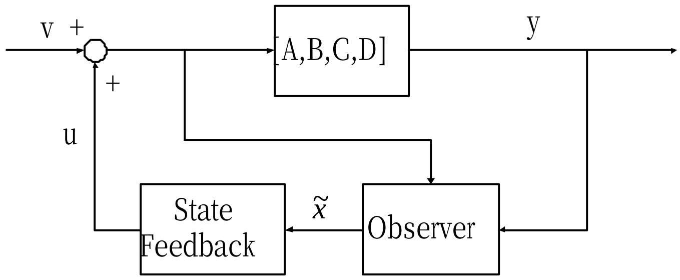
$$e(t) = \tilde{x}(t) - x(t)$$

$$\Downarrow$$

$$\dot{e}(t) = (A + lc^T)e(t).$$

Notice that the characteristic equation of $A + lc^T$ is the same as that of $A^T + cl^T$. Hence similar to the pole placement case, the poles associated with observer can be arbitrarily assigned on any location in the complex plane provided that (c, A) is observable. Indeed l that results in the desired poles can be computed from Bass-Gura or Ackermann's formula where (A^T, c) is used instead of (A, b).

Output Feedback



The state representation of the closed loop:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} + bk^T & bk^T \\ 0 & \boldsymbol{A} + lc^T \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} v(t)$$

$$y(t) = [c^T\ 0]\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}.$$

Notice that

$$det\begin{bmatrix} \boldsymbol{A} + bk^T & bk^T \\ 0 & \boldsymbol{A} + lc^T \end{bmatrix} = det(\boldsymbol{A} + bk^T)det(\boldsymbol{A} + bk^T).$$

Hence we have the following separation principle.

Separation Principle: the family of poles of the closed loop system is the union of those of state feedback system and state estimator.

Thanks to the separation principle, the static state feedback controller and asymptotic observer can be designed separately.

Notice that

$$G(s) = c^T(s\boldsymbol{I} - \boldsymbol{A} - bk^T)^{-1}b.$$

Hence the dynamics of observer doesn't show up in the external behavior due to the assumption that $e(0) = 0$.