Stochastic analysis of fractional order system: Block Pulse Technique

Pham Luu Trung Duong and Moonyong Lee*

School of display and chemical engineering, Yeungnam Univeristy, Gyeongsan, South Korea

(mynlee@yu.ac.kr*)

I. INTRODUCTION

In most engineering applications, one aims to solve physical problems by converting it into a deterministic mathematical model. This is a rough approximation of reality, as many physical input parameters describing the problem are fixed through this conversion. In reality, however, these parameters exhibit randomness with definite influences over behavior of the solution. Accordingly, it becomes increasingly important to quantify uncertainties associated with model predictions. Because of the "uncertain" nature of the uncertainty, the most dominant approach is to treat data uncertainty as random variables or random process.

The traditional statistical approach for stochastic analysis is the Monte Carlo (MC) method [5, 6]. With the brute force MC implementation, one first generates an ensemble of random realizations with each parameter drawn from its uncertainty distribution. Deterministic solvers are then applied to each member to obtain an ensemble of results. The ensemble of results is then post-processed to obtain the relevant statistical properties of the results, such as mean and standard deviation, as well as the probability density function (PDF). Since estimation of the variance converges with the inverse square root of the number of runs, the MC approach is computationally expensive [5, 8].

Polynomial Chaos [9, 10, 11] is another frequently used non-sampling techniques for stochastic analysis, however, it is known to fail for long-term integration and lose optimal convergence behavior leading to unacceptable error-levels even in simple stochastic differential equations [11].

Operational matrix is another modern approach to quantify uncertainty in system models. This technique, known as the spectral or operational matrix, is based on a finite-dimensional approximation of the mathematical model of a system using orthogonal expansions. The main characteristic of this technique is reduction of a system of differential equations into algebraic equations, thus greatly simplifying the problem. In this paper, the operational matrix is used to account for the influence of the random changes in the parameter of fractional order system on the statistical characteristics of its output when the disturbance is deterministic or stochastic.

II. FRACTIONAL ORDER SYSTEM

1. *Fractional integral and derivative*

Fractional integral Rieman Liouville integral of function $f(t)$ is defined by [1]

$$({}_aI^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau \tag{1}$$

Among several formulas of the generalized derivative, the most common used one is the Riemann-Liouville definition

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}}\,d\tau \tag{2}$$

where m is the integer satisfied $m-1<\alpha<m$

For the generalized integration and differentiation, the property of linearity, similarly to the integer case, is conserved.

2. *Fractional linear models*

Laplace transform of fractional order differentiation is defined as

$$? \;\frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} f^{k}(0^{+})s^{\alpha-1-k} \tag{3}$$

In particular, if the derivatives of the function $f(t)$ are all equal to 0 at, Eq. (3) can be rewritten as

$$? \ \frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha} F(s) \tag{4}$$

The fractional order linear system of single variablescan be defined as

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^{\beta_m} + ... + b_0 s^{\beta_0}}{s^{\alpha_n} + ... + a_0 s^{\alpha_0}} \tag{5}$$

The orders α_i, β_i are arbitrary real positive, $r(t)$ and $y(t)$ are respectively the input and output of systems.

3. Generalized operational matrix of block pulse functions for fractional integral

The generalization of operational matrix of integration to a positive real order α according to the Riemann–Liouville formula is the following [7]

$$(I_0^{\alpha} \Phi_N)(t) = P_{\alpha} \Phi_N(t) \tag{6}$$

where

$$P_{\alpha} = (\frac{T}{N})^{\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_N \\ 0 & f_1 & f_2 & \cdots & f_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \cdots & \cdots & f_1 \end{pmatrix} \tag{7}$$

with the elements given by recurrence relation

$$f_1 = 1 \ f_p = p^{\alpha+1} - 2(p-1)^{\alpha+1} + (p-2)^{\alpha+1} \ \ p = 2,3... \tag{8}$$

The operational matrix of derivative to a real order α may be calculated simply by

$$G_{\alpha} P_{\alpha} = I \tag{9}$$

III. STOCHASTIC ANALYSIS OF FRACTIONAL ORDER SYSTEM

The fractional order system in (5) can be rewritten in term of generalized operational matrix as follows

$$A_G = (I + a_{n-1} D_{\alpha_{n-1}} + ... + a_0 D_{\alpha_0})^{-1} (b_m D_{\beta_m} + ... + b_o D_{\beta_o}) \tag{10}$$

where $D_{\alpha_j} = G_{\alpha_j}{}^T$ and input and output of system related by

$$C^y = A_G C^r$$
$$y(t) = (C^y)^T \Phi^N(t) \quad r(t) = (C^y)^T \Phi^N(t) \tag{11}$$

where $\Phi^N(t)$ is family of block pulse functions.

If a_i, b_j are random variables (r.v), by utilizing the geometric series one may obtain the so- called stochastic fractional order matrix A^{st} as below

$$a_i = \overline{a}_i + a_i^r$$
$$b_j = \overline{b}_j + b_j^r \tag{12}$$

Where $\overline{a}_i, \overline{b}_j$ is the mean of r. v and a_i^r, b_j^r are random central component.

$$A = (I + A_X)^{-1} A_R = (I + \overline{A}_X + A_X{}^r)^{-1} A_R$$

$$A = A_0 \sum_{v=0}^{\infty} (-1)^v (A_X{}^r A_0)^v A_R \tag{13}$$

$$A = A_0 \sum_{j=0}^{m} \sum_{v=0}^{\infty} (-1)^v (\sum_{i=0}^{n-1} D_{\alpha_i} a_i^r A_0)^v D_{\beta_j} b_j$$

Consider the output and input signals in the form of Fourier series expansions (11) and the spectral characteristics of the output and input are linked by $C^y = AC^r$.

Thus, an equation for the output of stochastic systems is

$$Y_N(t) = \Phi^T(t)C^Y = \Phi^T(t)AC^l \quad (14)$$

where A is the stochastic matrix operator defined by (12).

Using the link between spectral characteristics of input and output, the mean of output signal for fractional order system can be calculated as

$$m^l{}_Y(t) = M[Y_l(t)] = M[\Phi^T(t)C^{m_Y}] = \Phi^T(t)M[C^Y] = \Phi^T(t)M[AC^r] \quad (15)$$

From the statistical independence of matrix A and column vector of coefficient expansion of input C^r

$$m^l{}_Y(t) = \Phi^T(t)C^{m_Y} = \Phi^T(t)M[A]M[C^r] = \Phi^T(t)\overline{A}C^{m_r} \quad (16)$$

Thus, the spectral characteristic of the mathematical expectations of the output and input signals of the stochastic system are related by

$$C^{m_Y} = \overline{A}C^{m_r} \quad (17)$$

Accordingly, the spectral characteristic of the mathematical expectation of the output signal is defined as a linear transformation of the spectral characteristic of the mathematical expectation input.

Deterministic matrix operator $\overline{A}$ is the expectation of random stochastic matrix operator A in Eq. (13), which can be expanded as

$$\overline{A} = M[A^{st}] = M\{A_0\sum_{j=0}^{m}\sum_{v=0}^{\infty}(-1)^v(\sum_{i=0}^{n-1}D_{\alpha_i}a_i^r A_0)^v D_{\beta_j}b_j\} = A_0\sum_{j=0}^{m}\sum_{v=0}^{\infty}(-1)^v M\{(\sum_{i=0}^{n-1}D_{\alpha_i}a_i^r A_0)^v D_{\beta_j}b_j\} \quad (18)$$

The stochastic moments of arbitrary-order for central random component $(a_i^r)^v$ in Eq. (18) may be calculated for each ν using the method mentioned in [12].

Equation (18) shows how the random parameters given in $\overline{A}$ affect the expectation of the output. The mathematical expectation of the output system, as determined by (17), (18), can be calculated with a desired accuracy that depends on the expectation of stochastic matrix operator, which in turn is determined by ν, the number of terms for approximation in (18).

The correlation function of the output stochastic system and its second central moment are next defined. By introducing the signal of the system in the form of Fourier series, the equation to define the second moment of output can be written as

$$\theta^l{}_{YY}(t_1,t_2) = M[Y_l(t_1)Y_l(t_2)] = M[\Phi^T(t_1)C^Y(C^Y)^T\Phi(t_2)] = \Phi^T(t_1)M[C^Y(C^Y)^T]\Phi(t_2) = \Phi^T(t_1)M[AC^r(C^r)^T A^T]\Phi(t_2) \quad (19)$$

Thus, (19) can take the form of

$$\theta^l{}_{YY}(t_1,t_2) = \Phi^T(t_1)M[AC^{\theta_{rr}}A^T]\Phi(t_2) \quad (20)$$

where $C^{\theta_{rr}}$ is the square matrix of the spectral characteristic of the second moment of the input of the system, which is determined using (21):

$$\theta^l{}_{rr}(t_1,t_2) = M[r_l(t_1)r_l(t_2)] = \Phi^T(t_1)M[C^r(C^r)^T]\Phi^T(t_2) = \Phi^T(t_1)M[C^{\theta_{rr}}]\Phi^T(t_2) \quad (21)$$

The covariance function or the second centered moment of the output system is defined as

$$K^l{}_{YY}(t_1,t_2) = M\{[Y_l(t_1) - m^l{}_Y(t_1)][Y_l(t_2) - m^l{}_Y(t_2)]\} = M[Y_l(t_1)Y_l(t_2)] - m^l{}_Y(t_1)m^l{}_Y(t_2)$$
$$= \theta^l{}_{YY}(t_1,t_2) - m^l{}_Y(t_1)m^l{}_Y(t_2) \quad (22)$$

where the first order moment is determined by (17), (18) and the second moment by (20).

The covariance function of the input signal is similarly associated with the second order moment

$$K^l{}_{rr}(t_1,t_2) = \theta^l{}_{rr}(t_1,t_2) - m^l{}_r(t_1)m^l{}_r(t_2) \quad (23)$$

Furthermore, the covariance function of the input signal can be expanded in terms of the orthonormal basis:

$$K^l{}_{rr}(t_1,t_2) = \Phi^T(t_1)C^{K_{rr}}\Phi(t_2) = \Phi^T(t_1)C^{\theta_{rr}}\Phi(t_2) - \Phi^T(t_1)C^{m_r}(C^{m_r})^T\Phi(t_2) \quad (24)$$

Thus, the spectral characteristic of the moments of input signal are related by

$$C^{K_{rr}} = C^{\theta_{rr}} - C^{m_r}(C^{m_r})^T \quad (25)$$

Equation (20) can thus be rewritten as follows

$$\theta^l{}_{YY}(t_1,t_2) = \Phi^T(t_1)M\{A[C^{K_{rr}} + C^{m_r}(C^{m_r})^T]A^T\}\Phi(t_2) \quad (26)$$

Taking into account (22) and (26), the following equation is obtained for the covariance function of the output stochastic system:

$$K^l{}_{YY}(t_1,t_2) = \Phi^T(t_1)C^{K_{YY}}\Phi(t_2) = \Phi^T(t_1)M\{A[C^{K_{rr}} + C^{m_r}(C^{m_r})^T]A^T\}\Phi(t_2) - \Phi^T(t_1)C^{m_Y}(C^{m_Y})^T\Phi(t_2) \quad (27)$$

or

$$C^{K_{YY}} = C^{\theta_{YY}} - C^{m_Y}(C^{m_Y})^T = M\{A[C^{K_{rr}} + C^{m_r}(C^{m_r})^T]A^T\} - C^{m_Y}(C^{m_Y})^T \quad (28)$$

where A is the stochastic operational matrix defined by (20).

Equation (28) gives the relation between the spectral characteristics of the covariance function of the output and input signal, and the mathematical expectations of the output and input signal.

IV. CASE STUDIES

Consider fractional order system $\frac{K}{s^{1.4}+1}$ where K is random variable. Several simulation examples with different types of random gain and input are used to validate the correctness of the method. The stochastic signals used in the simulation are all Gaussian random process. All simulation parameters are described in Table 1. Statistical characteristics of closed loop systems for both regulatory and servo problems are shown in Figs. 1, which delineate the consistency between the operational matrix and Monte Carlo method. A computer, with AMD Phenom II X3 2.81 GHz 2GB RAM, was used for the test with simulation times also shown in Table 1.

Case	**Monte Carlo**		**Operational Matrix**
1) $K \in U(0.5,1.5)$	N of S: 2000	Comp. time 6.14 s.	Comp .time 1.2 s.
2) $K \in N(1,0.01)$	N of S :2000	Comp. time 5.8 s.	Comp .time 1.2 s.
3) $K \in U(0.5,1.5)$ $M_x = 1(t); K_{rr} = 0.01e^{-5\|t_1-t_2\|}$	N of S: 8400	Comp. time 158 s.	Comp .time 1.2 s.
4) $K \in N(1,0.01)$ $M_x = 1(t); K_{rr} = 0.01e^{-5\|t_1-t_2\|}$	N of S: 8400	Comp. time 155 s	Comp .time 1.2 s.

Table 1 Simulation parameter and computational time

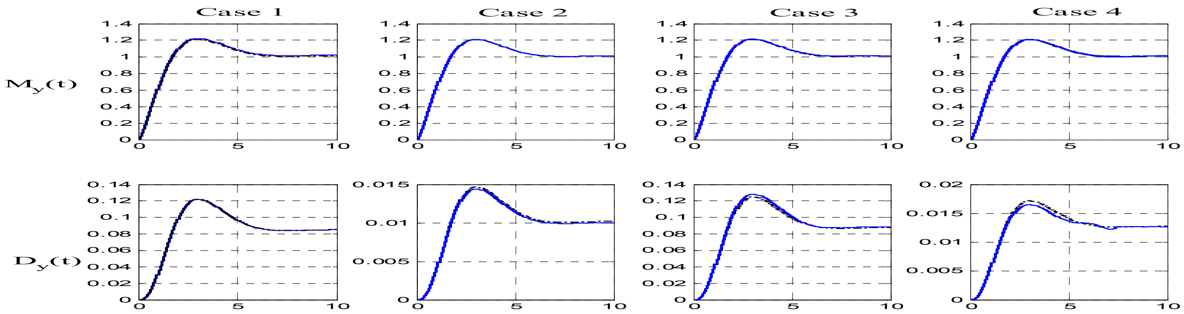

Figure 1 Statistical characteristic of output for fractional order system. Black:operational matrix; Blue Monte-Carlo

V. CONCLUSIONS

In this work, a statistical analysis for fractional order system was studied. It is shown that the use of Block Pulse Operational matrix method drastically reduces a computation time with a desired accuracy over that by the traditional Monte-Carlo method. Simulation examples have shown that the method gives accurate results for prediction statistical characteristic of fractional order system with random parameter under exciting by both deterministic and stochastic input.

REFERENCES

[1] K. A. Puvkov, N. D. Egupov, A. M. Makarenkov, *Theory and Numerical Methods for Studying Stochastic Systems*, Moscow, Fizmatlits, 2003[in Russian].
[2] J. S. Liu, *Monte Carlo Strategies in Scientific Computing*, Springer-Verlag, 2001.
[3] N. Wiener, The homogeneous chaos, *American Journal of Mathematics*, **60**, 1938 897–936.
[4] R. G. Ghanem, P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Dover publications, 1991.
[5] D. Xiu, G. R. Karniadis, Modeling uncertainty in steady state diffusion problem via generalized polynomial chaos, *Computer method in applied mechanics and Engineering* , **191 (43)** , 2002, 4927-4948.
[6] K. A. Puvkov, N. D. Egupov, A. M. Makarenkov, *Theory and Numerical Methods for Studying Stochastic Systems*. Moscow, Fizmatlits, 2003[in Russian].
[7] C. H .Wang, On the generalization of block pulse operational matrices for fractional calculus application, J. Franklin Inst , 1983, 315(2), 91-102.
[8] S. S. Isukapalli, A. Roy, P. G. Georgopoulos, Efficient sensitivity/uncertainty analysis using the combined stochastic response surface method, and automated differentiation: application to environmental and biological systems, *Risk Analysis* , **20 (5)** , 2000, 591–602
[9] M. A. Tatang, Direct incorporation of uncertainty in chemical and environmental engineering systems, Ph.D. Thesis, Massachusetts Institute of Technology, 1995.
[10] A. Sandu, J.G.Verwer, J.G. Blom, E.J. Spee, G.R. Carmichael, F.A. Potra, Stiff ODE solvers for atmospheric chemistry problems II: Rosenbrock, solvers, *Atmospheric Environment* **31** (1997) 3459–3472.
[11] H. Cheng, A. Sandu , Efficient uncertainty quantification with the polynomial chaos method for stiff systems, *Mathematics and Computers in Simulation* , **79** (2009) 3278–3295.
[12] J. A. Gubner , Probability and random process for electrical and computer engineering , Cambridge University Press, 2006.