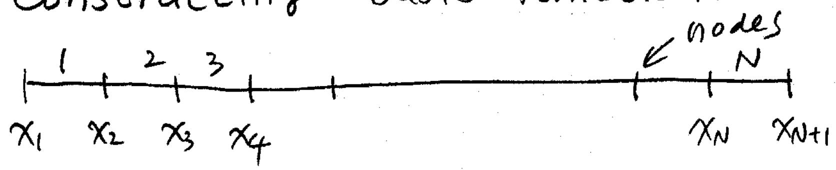
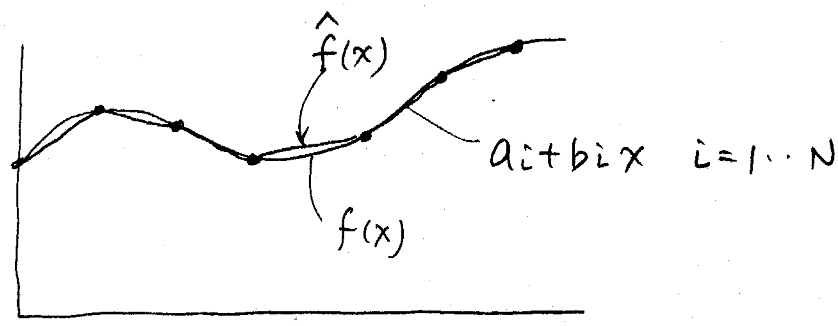
5. Finite Element Method

(1) Basis Function

<1) Linear Basis function

Constructing basis function





Interpolation problems. :

I know $f(x)$ and I want to find (a_i, b_i)
2 unknowns.

$(N+1)$ conditions $f(x_i) = \hat{f}(x_i)$
interpolated value

$(N-1)$ matching conditions
continuity (primitive form)

Cardinal form of the basis.

$$f(x) \cong \hat{f}(x) \equiv \sum_{i=1}^{N+1} \alpha_i \, \Phi^i(x)$$

basis function

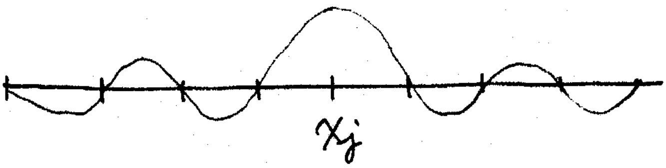
Φ^i satisfies

$\Phi^i(x_j) = \delta_{ij}$

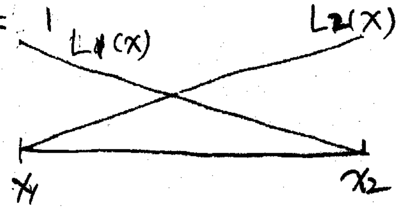
$\{\alpha_i\}$ are the values of function at each pt

$$f(x_j) = \hat{f}(x_j) = \sum_{i=1}^{M} \alpha_i \delta_{ij} = \alpha_j$$

Global Lagrangian interpolation function $L_j(x)$

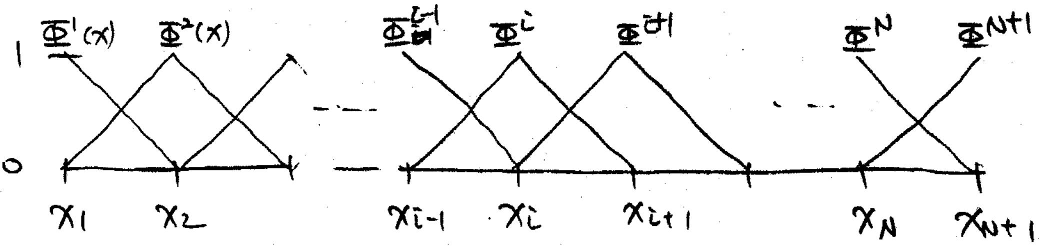


For $N = 1$



$$L_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad L_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Local Lagrangian basis function $\Phi^i(x)$



$\Phi^i(x)$ Roof top function

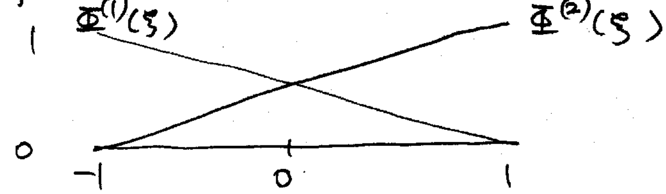
Hat function

Chapeau function

Linear Basis function

Systematic calculation of $\Phi^i(x)$

Define a unit element



$$\Phi^{(1)}(\xi) = a_1 + b_1\xi = \frac{1-\xi}{2}$$

$$\Phi^{(2)}(\xi) = a_2 + b_2\xi = \frac{1+\xi}{2}$$

Transform every element to unit element

$$x_k \le x \le x_{k+1} \longrightarrow -1 \le \xi \le 1$$

$$x(\xi) = a + b\xi$$

$$\boxed{x(\xi) = \sum_{i=1}^{2} c_i \, \Phi^{(i)}(\xi)}$$ Isoparametric mapping.

Find $\{c_i\}$ $\quad x_k \longleftrightarrow x_{(1)}$

$\quad x_{k+1} \longleftrightarrow x_{(2)}$

$$x_{(1)} = \sum_{i=1}^{2} c_i \, \Phi^{(i)}(-1) = c_1$$

$$x_{(2)} = \sum_{i=1}^{2} c_i \, \Phi^{(i)}(+1) = c_2$$

$$\longrightarrow x(\xi) = \sum_{i=1}^{2} x_{(i)} \, \Phi^{(i)}(\xi)$$

x vs. ξ

$$\xi = -1 + 2\,\frac{x - x_k}{x_{k+1} - x_k}$$

$$x = x_k + (x_{k+1} - x_k)\,\frac{1+\xi}{2}$$

$$\Phi^{(1)}(\xi) = \qquad = \Phi^{k}(x)$$

$$\Phi^{(2)}(\xi) = \Phi^{k+1}(x)$$

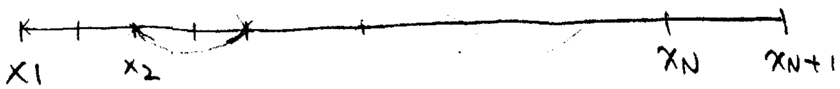
$$\frac{d\Phi^k}{dx} = \frac{d\Phi^k}{d\xi}\frac{d\xi}{dx} = \frac{d\Phi^{(1)}}{d\xi}\frac{d\xi}{dx}$$

$$= \frac{d\Phi^{(1)}}{d\xi}\,\frac{2}{x_{k+1} - x_k}$$

length ratio.

2) Quadratic Basis function

N elements



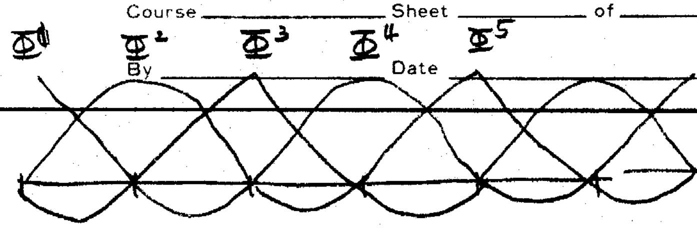
$$\hat{f}(x) = a_i + b_i x + c_i x^2 \qquad i = 1 \cdots N$$

3 N a. unknown

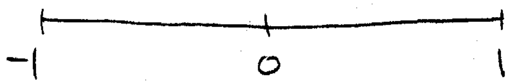
N-1 continuity

2N+1 degree of freedom.

Cardinal Basis



Local element

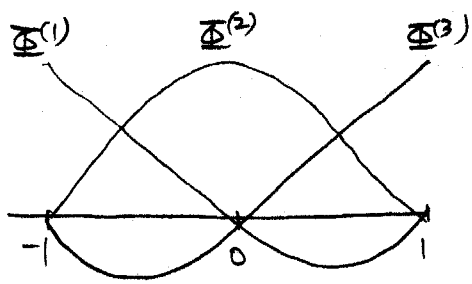


unit element

Construct Approx function associated with each node

$$\Phi^{(i)}(\xi) = a_i + b_i \xi + c_i \xi^2 \qquad i = 1, 2, 3$$

Interpolation condition : $\Phi^{(i)}(\xi_j) = \delta_{ij}$



$$\Phi^{(1)} = \frac{\xi(\xi-1)}{2}, \quad \Phi^{(2)} = 1 - \xi^2, \quad \Phi^{(3)} = \frac{\xi(\xi+1)}{2}$$

Approximation

$$\hat{f}(x) = \sum_{i=1}^{2N+1} f(x_i)\, \Phi^i(x)$$

Hermite Cubic Basis.

Assume a cubic approximation.

$$\hat{f}(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

Lagrangian
4N unknown
N-1 continuity
3N+1 D of freedom
Lagrangian

Hermite
4N unknowns
N-1 continuity
N-1 derivative continuity
2N+2 interpolation condition.

		Order of Poly	# of DOF (N elems)	Accuracy
Lagrangian	Linear	1	N+1	$O(h^2)$
	Quad	2	2N+1	$O(h^3)$
	Cubic	3	3N+1	$O(h^4)$
Hermite	Cubic	3	2N+2	$O(h^4)$

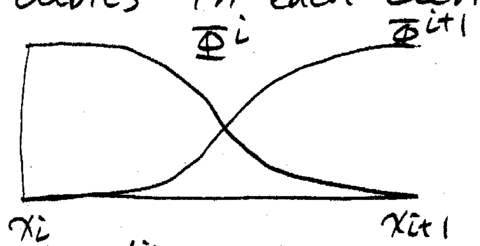
Fewer unknowns than Lagrangian.

Cardinal Form

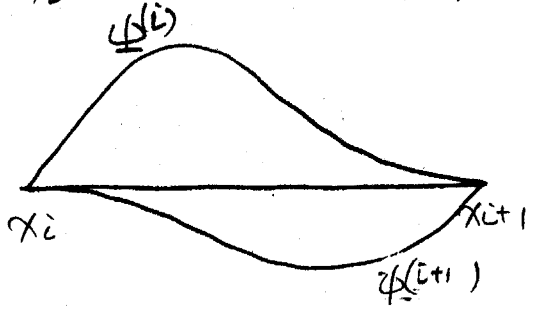
$$\hat{f}(x) = \underbrace{\sum_{i=1}^{N+1} \alpha_i \Phi^i(x) + \sum_{i=1}^{N+1} \beta_i \Psi^i(x)}_{2N+2 \text{ DOF}}$$

Conditions

$\Phi^i(x)$, $\Psi^i(x)$ are cubics in each element.



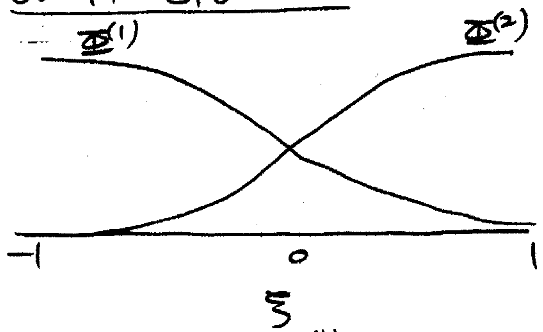
1. $\Phi^i(x_j) = \delta_{ij}$
2. $\frac{d\Phi^i}{dx}(x_j) = 0$
3. $\Psi^i(x_j) = 0$
4. $\frac{d\Psi^i}{dx}(x_j) = \delta_{ij}$



$f(x_j) = \alpha_j$

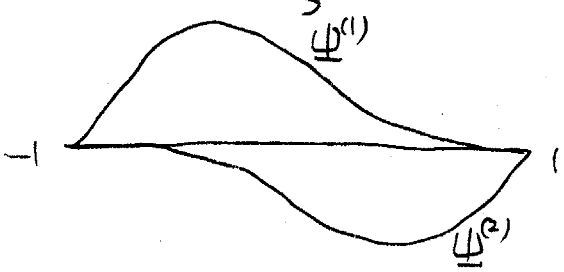
$\left.\frac{df}{dx}\right|_{x_j} = \beta_j$

Unit Element



$$\Phi^{(1)}(\xi) = -\frac{1}{4}((1-\xi)^2\xi$$

$$\Phi^{(2)}(\xi) = \frac{1}{4}(2-\xi)(\xi+1)^2$$



$$\Psi^{(1)}(\xi) = \frac{1}{4}(\xi-1)^2(\xi+1)$$

$$\Psi^{(2)}(\xi) = \frac{1}{4}(\xi-1)(\xi+1)^2$$