2.2 Linear Vector Space

2.2.1 Linear vector space

Def A set $\mathcal{V}$ is defined as a **linear vector space** over a scalar field $\mathcal{F}$, if the operation of addition is defined for members of $\mathcal{V}$ and multiplication of elements of $\mathcal{V}$ by an element of the field $\mathcal{F}$ is defined. That is,

1. If $\underline{x} \in \mathcal{V}, \underline{y} \in \mathcal{V}$, then $\underline{x} + \underline{y} \in \mathcal{V}$ and unique.
2. If $\underline{x} \in \mathcal{V}, \alpha \in \mathcal{F}$, then $\alpha\underline{x} \in \mathcal{V}$.

These operations must obey the following properties.

1. $\underline{x} + \underline{y} = \underline{y} + \underline{x}$
2. $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$
3. There is a unique vector $\underline{0}$ in $\mathcal{V}$, called zero vector, such that

   $$\underline{x} + \underline{0} = \underline{x} \text{ for all } \underline{x} \in \mathcal{V}$$

4. For each vector $\underline{x} \in \mathcal{V}$, there is a unique vector $-\underline{x}$ such that

   $$\underline{x} + (-\underline{x}) = \underline{0}$$

5. $1\underline{x} = \underline{x}$ for all $\underline{x} \in \mathcal{V}$.
6. If $\alpha, \beta \in \mathcal{F}$, then $\alpha(\beta\underline{x}) = (\alpha\beta)\underline{x}$ for all $\underline{x} \in \mathcal{V}$.
7. If $\alpha \in \mathcal{F}, \underline{x}, \underline{y} \in \mathcal{V}$, then $\alpha(\underline{x} + \underline{y}) = \alpha\underline{x} + \alpha\underline{y}$
8. If $\alpha, \beta \in \mathcal{F}, \underline{x} \in \mathcal{V}$, then $(\alpha + \beta)\underline{x} = \alpha\underline{x} + \beta\underline{x}$

Example

- $\Re^n$: Collection of all n-dimensional vectors with real components
- $\mathcal{C}^n$: Collection of all n-dimensional vectors with complex components
- $\Re^{m\times n}$: Collection of all $m \times n$ matrices with real components

2.2.2 Linearly independent set

Def A set of vectors $\{\underline{x}_i\} \in \Re^n$ is said to be **linearly dependent** if there exists n scalars $\{\alpha_i\} \in \Re$, such that

$$\alpha_1\underline{x}_1 + \alpha_2\underline{x}_2 + \cdots + \alpha_n\underline{x}_n = 0$$

where not all of the $\{\alpha_i\}$ are zero.

If no such set of scalars exists, the vectors $\{\underline{x}_i\}$ are **linearly independent**.

For linearly dependent set, one of the vectors can be written as a **linear combination** of the other elements in the space.

$$\underline{x}_1 = -\frac{1}{\alpha_1}\sum_{j=2}^{n}\alpha_j\underline{x}_j$$

Example in $\Re^3$

Most obvious example of a linearly independent set

$$\begin{aligned} \underline{e}_1^T &= (1,0,0) \\ \underline{e}_2^T &= (0,1,0) \\ \underline{e}_3^T &= (0,0,1) \end{aligned}$$

These are called base vectors or basis.

Example in $\Re^{2\times 2}$

Basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

2.2.3 Subspace

Def Let $\mathcal{W}$ be a non-empty subset of $\Re^n$. Then, $\mathcal{W}$ is a subspace of $\Re^n$ if

1. $\underline{x}, \underline{y} \in \mathcal{W}$ implies $(\underline{x} + \underline{y}) \in \mathcal{W}$

2. $\underline{x} \in \mathcal{W}$ and $\alpha \in \Re$ implies $\alpha\underline{x} \in \mathcal{W}$.

Ex 1: All vectors of the form $\underline{x}^T = (\alpha_1, \alpha_2, \dots, \alpha_r, 0, \dots, 0)$. (last $n - r$ elements of the vector are zero).

Ex 2: All 2×2 matrices that are symmetric. $\mathcal{W}$ is a subset of $\Re^{2\times 2}$.

Theorem Let $\mathcal{A}$ be a set of vectors in $\Re^n$ and let $\mathcal{W}$ be the set of all linear combination of elements in $\mathcal{A}$, Then, $\mathcal{W}$ is a subspace of $\Re^n$ and each elements in $\mathcal{W}$ is written as

$$\underline{y} = \alpha_1\underline{x}_1 + \alpha_2\underline{x}_2 + \cdots + \alpha_m\underline{x}_m$$

where $\{\underline{x}_i\} \in \mathcal{A}$, can $\{\alpha_i\} \in \Re$.

Def Let $\mathcal{W} \subset \Re^n$ be a subspace and let $\mathcal{B}$ a set of vectors from $\Re^n$. Then $\mathcal{B}$ is a basis for $\mathcal{W}$ if

1. the elements of $\mathcal{B}$ are linearly independent
2. $\mathcal{B}$ spans or generates $\mathcal{W}$, that is each vector in $\mathcal{W}$ is a linear combination of elements of $\mathcal{B}$.

Def If $\mathcal{W}$ is a subspace of $\Re^n$, the dimension of $\mathcal{W}$ is the number of elements in a basis for $\mathcal{W}$.

Theorem The dimension of a subspace $\mathcal{W}$ is unique even though the components of the basis are not.

2.2.4 Linear dependence

Linear dependence of a set of n vectors:
For $(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$ (each $\underline{a}_i \in \Re^n$) and for a set of numbers $\{\alpha_i\} \in \Re$,

$$\underline{a}_1\alpha_1 + \underline{a}_2\alpha_2 + \cdots + \underline{a}_n\alpha_n = 0$$

where each vector $\underline{a}_i$ can be written as

$$\underline{a}_i^T = (a_{1i}, a_{2i}, \dots, a_{ni})$$

Then,

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \alpha_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} \alpha_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \alpha_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$\underline{\underline{A}}\, \underline{\alpha} = \underline{0}$$

That is, linear dependence $\equiv$ nontrivial solution (other than $\underline{\alpha} = \underline{0}$) of homogeneous linear equation

case I If rank($\underline{\underline{A}}$) $= n$, the inverse matrix is well-defined and $\underline{\alpha} = \underline{0}$ is the unique solution $\longrightarrow \{\underline{a}_i\}$ is linearly independent

case II For the matrix with rank($\underline{\underline{A}}$) $= n - 1$
Assume an ordering

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

For submatrix $\underline{\underline{\tilde{A}}}$,

$$\underline{\underline{\tilde{A}}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix}$$

$\det(\underline{\underline{\tilde{A}}}) \neq 0 \longrightarrow \underline{\underline{\tilde{A}}}^{-1}$ exists.
Neglecting the last row,

$$\underline{\underline{\tilde{A}}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = -\alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1,n} \end{bmatrix}$$

$\longrightarrow$ Assume α_n, then we can determine $\alpha_1, \ldots, \alpha_{n-1}$.

$\longrightarrow$ Then,

$$\underline{a}_1\alpha_1 + \cdots + \underline{a}_{n-1}\alpha_{n-1} + \underline{a}_n\alpha_n = 0$$

$\longrightarrow$ $\underline{a}_n$ is a linear combination of the first $n-1$ vectors.
That is, $\{\underline{a}_i\}$ is linearly dependent

2.3 Linear Transformation and Linear Equation Sets

2.3.1 Linear transformation

$n \times m$ matrix
- transformation that takes an element of $\Re^m$ into an element of $\Re^n$.
- has the properties of linear transformation

Def A linear transformation from $\Re^m$ to $\Re^n$ is a matrix $\underline{\underline{A}} : \Re^m \rightarrow \Re^n$ such that

$$\underline{\underline{A}}(\alpha\underline{x} + \beta\underline{y}) = \alpha\underline{\underline{A}}\,\underline{x} + \beta\underline{\underline{A}}\,\underline{y}$$

where $\underline{x}, \underline{y} \in \Re^m, \alpha, \beta \in \Re$.

2.3.2 Range and null space

Def If $\underline{\underline{A}} \in \Re^{n\times m}$, the subspace of $\Re^n$ spanned by the columns $\{\underline{a}_i\}, i = 1, \ldots, m$ of $\underline{\underline{A}}$ is called the **range** of $\underline{\underline{A}}$ and is written as $\mathcal{R}(\underline{\underline{A}})$.

$$\dim(\mathcal{R}(\underline{\underline{A}})) = \text{number of linearly independent columns of } \underline{\underline{A}}$$

Def For $\underline{\underline{A}} \in \Re^{n\times m}$, the **null space** of $\underline{\underline{A}}, \mathcal{N}(\underline{\underline{A}})$, is the subspace of $\Re^m$ formed by all vectors, $\underline{x}$ such that $\underline{\underline{A}}\,\underline{x} = 0$.

$$\dim(\mathcal{N}(\underline{\underline{A}})) = \text{dimension of the basis of } \mathcal{N}(\underline{\underline{A}})$$

2.3.3 Homogeneous equation set

For homogeneous equation set $\underline{\underline{A}}\,\underline{x} = \underline{0}$, following theorem holds.

Theorem If $\underline{\underline{A}} \in \Re^{n\times m}$, the null space of $\underline{\underline{A}}, \mathcal{N}(\underline{\underline{A}})$, has $\dim(\mathcal{N}(\underline{\underline{A}})) = m - r$ where $r = \text{rank}(\underline{\underline{A}})$. If $r = m$, then $\underline{x} = \underline{0}$ is the only solution.

2.3.4 Nonhomogeneous equation set

Theorem Let $\underline{\underline{A}} \in \Re^{n\times m}$ and $\underline{\underline{A}} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_m)$ where $\underline{a}_i \in \Re^n$.
Then, $\underline{\underline{A}}\,\underline{x} = \underline{b}$, where $\underline{b} \in \Re^n$, has a solution $\underline{x} \in \Re^m$ (*i.e.*, is soluble) iff $\dim(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_m, \underline{b})$ is equal to $\dim(\mathcal{R}(\underline{\underline{A}}))$. That is $\underline{b}$ is a linear combination of $\{\underline{a}_i\}$.

Theorem Let $\underline{\underline{A}} \in \Re^{n\times m}$. If $\underline{b} \in \mathcal{R}(\underline{\underline{A}})$ and $\dim(\mathcal{R}(\underline{\underline{A}})) = m$, then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ possesses a unique solution.

Theorem Let $\underline{\underline{A}} \in \Re^{n\times m}$ and $\underline{b} \in \mathcal{R}(\underline{\underline{A}})$. If $\text{rank}(\underline{\underline{A}}) = r < m$, then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has the general solution $\underline{x} = \underline{x}_o + \underline{z}$ where $\underline{x}_o$ is a particular solution of $\underline{\underline{A}}\,\underline{x} = \underline{b}$ and $\underline{z}$ is any solution of the corresponding problem $\underline{\underline{A}}\,\underline{x} = \underline{0}$. *i.e.* $\underline{z} \in \mathcal{N}(\underline{\underline{A}})$ where $\dim(\mathcal{N}(\underline{\underline{A}})) = m - r$.