2.4 Algebraic Eigenvalue Problem

Algebraic eigenvalue problem is to find the pair of $(\underline{x}, \lambda)$ for

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}, \quad \underline{x} \neq \underline{0}, \ \underline{\underline{A}} \in \Re^{n\times n} \tag{2.1}$$

It is noted that

1. Eigenvalue problem is nonlinear. (Two unknowns λ and $\underline{x}$ are multiplied.)
2. The problem seems to be underspecified (n equations and $n+1$ unknowns). We don't expect unique solution of Eq. (2.1).

From Eq. (2.1)

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

EVP defines a homogeneous equation set.

Non-trivial solution $\underline{x}$ only if

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 = \mathcal{P}(\lambda)$$

where $\mathcal{P}(\lambda)$ is a characteristic polynomial with n-th order.

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

There is a term in $\mathcal{P}(\lambda)$

$$\prod_{i=1}^{n}(a_{ii} - \lambda) \rightarrow \lambda^n$$

and

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \mathcal{P}_n(\lambda)$$

From $\mathcal{P}_n(\lambda) = 0$, n roots of either real of complex conjugate.

- Eigenvectors $\{\underline{x}_i\}$ corresponding to $\{\lambda_i\}$ are determined only to within a multiplicative constant.

$$\begin{aligned} \underline{\underline{A}}\,\underline{x} &= \lambda \underline{x} \\ \underline{\underline{A}}(c\underline{x}) &= \lambda(c\underline{x}) \end{aligned}$$

Def $\mathcal{C}^n$: Vector space composed of all n-dimensional vectors with complex elements

$$\Re^n \subset \mathcal{C}^n$$

I. Eigenvectors of $\underline{\underline{A}}$ are linearly independent.

II. Adjoint Eigenvalue Problem
For $\underline{\underline{A}} \in \Re^{n \times n}$

$$\underline{\underline{A}}^T \underline{y} = \eta \underline{y} \quad \underline{y} \neq 0$$

$\{\eta_i\}$: adjoint eigenvalues
$\{\underline{y}\}$: adjoint eigenvectors
The two eigenvalue problems

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}, \quad \underline{\underline{A}}^T \underline{y} = \eta \underline{y}$$

have same eigenvalues.

III. Biorthogonality
For eigenvalue problem

$$\underline{\underline{A}}\,\underline{x}_j = \lambda_j \underline{x}_j$$

premultiplying $\underline{y}_i$ on both sides gives

$$\underline{y}_i^T \underline{\underline{A}}\, \underline{x}_j = \lambda_j \underline{y}_i^T \underline{x}_j \tag{2.2}$$

For adjoint eigenvalue problem

$$\underline{\underline{A}}^T \underline{y}_i = \lambda_i \underline{y}_i$$

Transpose gives

$$\underline{y}_i^T \underline{\underline{A}} = \lambda_i \underline{y}_i^T$$

postmultiplying $\underline{x}_j$ on both sides gives

$$\underline{y}_i^T \underline{\underline{A}}\, \underline{x}_j = \lambda_i \underline{y}_i^T \underline{x}_j \tag{2.3}$$

Subtracting Eq. (2.2) from Eq. (2.3) gives

$$0 = (\lambda_i - \lambda_j) \underline{y}_i^T \underline{x}_j$$

If $i \neq j$, then $\lambda_i \neq \lambda_j$, and

$$\underline{y}_i^T \underline{x}_j = 0 \tag{2.4}$$

This relation is called **biorthogonality**.

When $\underline{\underline{A}}$ is symmetric, then $\underline{\underline{A}} = \underline{\underline{A}}^T$ and $\{\underline{x}_i\} = \{\underline{y}_i\}$. Then Eq. (2.4) becomes

$$\underline{x}_i^T \underline{x}_j = 0$$

This relation is called **orthogonality**.

Theorem Solvability of $\underline{\underline{A}}\, \underline{x} = \underline{b}$ or **Fredholm's Alternative**

1. If $\underline{\underline{A}}\, \underline{x} = \underline{0}$ has only the trivial solutions then $\underline{\underline{A}}^{-1}$ exists and $\underline{\underline{A}}^{-1} \underline{b}$ is unique. $\dim[\mathcal{N}(\underline{\underline{A}})] = 0$.

2. If $\underline{\underline{A}}\,\underline{x} = 0$ has non-trivial solutions, so does $\underline{\underline{A}}^T\,\underline{y} = \underline{0}$.
   Assume $\dim[\mathcal{N}(\underline{\underline{A}})] = k < n$, then $\dim[\mathcal{N}(\underline{\underline{A}}^T)] = k$.

3. The necessary and sufficient condition for solving $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is that $\underline{y}_i^T\underline{b} = 0$ for $\underline{y}_i$ satisfying $\underline{\underline{A}}^T\,\underline{y}_i = \underline{0}$. This means $\underline{b}$ is orthogonal to all solutions of $\underline{\underline{A}}^T\,\underline{y} = \underline{0}$.

2.5 Norms for Vectors and Matrices

Norm: size of a vector or a size of the difference between two vectors.

Def A norm on $\Re^n$ is any real-valued function $\|\cdot\|$ $\Re^n \to \Re$ such that

1. If $\underline{x} \neq \underline{0}$, $\|\underline{x}\| > 0$
2. $\|\alpha \underline{x}\| = |\alpha| \|\underline{x}\|$, $\alpha \in \Re$
3. $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

Example :

1. ℓ_1-norm

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

2. ℓ_2-norm

$$\begin{aligned} \|\underline{x}\|_2 &= \left[\sum_{i=1}^{n} x_i^2 \right]^{1/2} \\ &= (\underline{x}^T \underline{x})^{1/2} \end{aligned}$$

3. ℓ_∞-norm

$$\|\underline{x}\|_\infty = \max\{|x_i|, i = 1, \ldots, n\}$$

Theorm Schwartz Inequality

For $\underline{x}, \underline{y} \in \Re^n$,

$$|\underline{x}^T \underline{y}| \leq \|\underline{x}\|_2 \|\underline{y}\|_2$$

Theorem Let $\|\cdot\|_u$ and $\|\cdot\|_v$ be two norms for $\Re^n$. Then there are positive constants a and b such that

$$a\|\underline{x}\|_u \leq \|\underline{x}\|_v \leq b\|\underline{x}\|_u \text{ for all } \underline{x} \in \Re$$

Example For $\|\underline{x}\|_\infty, \|\underline{x}\|_1$

$$\|\underline{x}\|_\infty \leq \|\underline{x}\|_1 \leq n\|\underline{x}\|_\infty$$

Matrix Norm .

Defn For $\|\underline{\underline{A}}\| \in \Re^{m\times n}$, $\|\cdot\| : \Re^{m\times n} \to \Re$ is a matrix norm if

1. (a) $\underline{\underline{A}} \neq \underline{\underline{0}}$ implies $\|\underline{\underline{A}}\| > 0$.
   (b) $\underline{\underline{A}} = \underline{\underline{0}}$ implies $\|\underline{\underline{A}}\| = 0$.
2. $\|\alpha\underline{\underline{A}}\| = |\alpha|\|\underline{\underline{A}}\|$.
3. $\|\underline{\underline{A}} + \underline{\underline{B}}\| \leq \|\underline{\underline{A}}\| + \|\underline{\underline{B}}\|$.
4. $\|\underline{\underline{A}}\,\underline{\underline{B}}\| \leq \|\underline{\underline{A}}\|\,\|\underline{\underline{B}}\|$. (This is only for $n \times n$ matrix)

Example .

1. Frobenius norm

$$\|\underline{\underline{A}}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

2. Maximum of column sums

$$\|\underline{\underline{A}}\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|$$

3. Maximum of row sums

$$\|\underline{\underline{A}}\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|$$

Defn Matrix and vector norms are consistent if they satisfy the inequality

$$\|\underline{\underline{A}}\,\underline{x}\| \leq \|\underline{\underline{A}}\|\,\|\underline{x}\|$$

Consistent matrix norm Construction of consistent matrix norm

$$\|\underline{\underline{A}}\| = \max_{\underline{x}\in\Re^n} \frac{\|\underline{\underline{A}}\,\underline{x}\|}{\|\underline{x}\|}, \text{ where } \underline{\underline{A}} \in \Re^{m\times n}$$

For arbitrary $\underline{x}$,

$$\frac{\|\underline{\underline{A}}\,\underline{x}\|}{\|\underline{x}\|} \leq \|\underline{\underline{A}}\|$$