2.10 Solution Tracking

We want to solve

$$\underline{R}(\underline{x}; p) = \underline{0}$$

for $\underline{x}$ by varying parameter p.

Then, $\underline{x} = \underline{x}(p)$ is a solution curve. (See Figure 2.1)

Taylor series around $\underline{x}(p_o)$ (known)

$$\underline{x}(p_o + \delta p) = \underline{x}(p_o) + \left.\frac{\partial \underline{x}}{\partial p}\right|_{p_o} \delta p + \mathcal{O}(\delta p)^2$$

This is called analytic continuation.
How do we obtain $(\frac{\partial \underline{x}}{\partial p})_{p_o}$, which is a tangent to the curve?
Taylor series for $\underline{R}(\underline{x}(p_o + \delta p))$

$$\underline{R}(\underline{x}(p_o + \delta p)) = \underline{R}(\underline{x}(p_o)) + \left.\frac{\partial \underline{R}}{\partial \underline{x}}\right|_{\underline{x}(p_o)} \left(\frac{\partial \underline{x}}{\partial p}\right)_{p_o} \delta p + \left(\frac{\partial \underline{R}}{\partial p}\right)_{\underline{x}_o} \delta p$$

It is rewritten as

$$\underline{R}(\underline{x}(p_o + \delta p)) = \underline{R}(\underline{x}(p_o)) + \underline{\underline{J}}(\underline{x}_o) \left(\frac{\partial \underline{x}}{\partial p}\right)_{p_o} \delta p + \left(\frac{\partial \underline{R}}{\partial p}\right)_{\underline{x}_o} \delta p$$

Since $\underline{\underline{R}}(\underline{x}) = 0$ along the solution curve

$$\underline{\underline{J}}(\underline{x}_o) \left(\frac{\partial \underline{x}}{\partial p}\right)_{p_o} = -\left(\frac{\partial \underline{R}}{\partial p}\right)_{p_o}$$

How much work?
Assume we use Newton's method to compute $\underline{x}(p_o)$,

$$\underline{\underline{J}}(\underline{x}_o) = \underline{\underline{L}}\,\underline{\underline{U}}$$

Then, the work to compute $(\frac{\partial \underline{x}}{\partial p})_{p_o}$ is trivial.

What controls success?

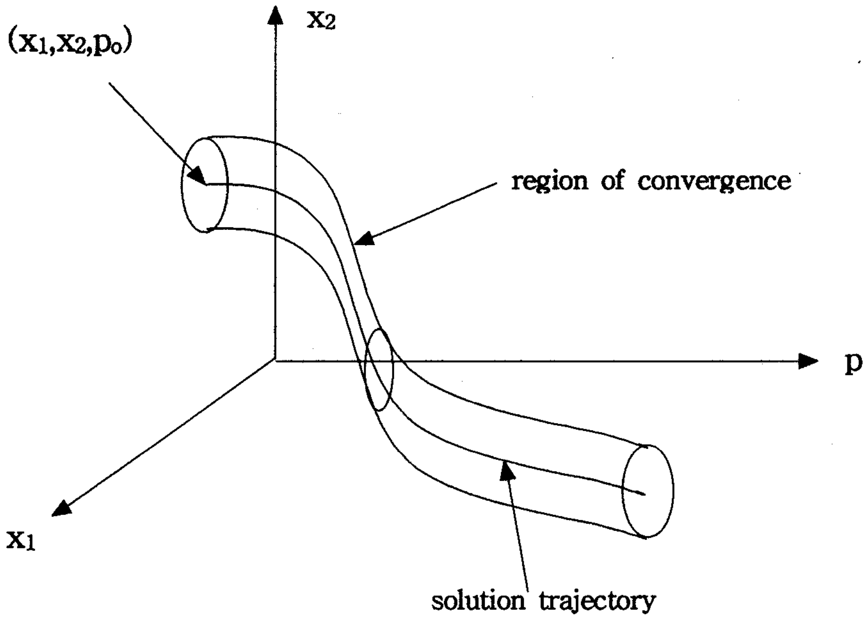

Figure 2.1. Solution trajectory.

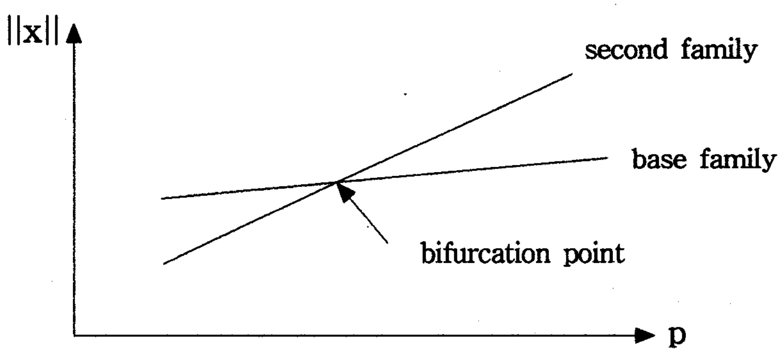

Figure 2.2. Bifurcation.

- δp
  proved convergence when $\delta p \to 0$.

- $\underline{\underline{J}}$

  1. $\det(\underline{\underline{J}}) \neq 0$: unique solution $\underline{x}_P$ (regular point)
  2. $\det(\underline{\underline{J}}) = 0$: No solution or multiple solutions (singular point or critical point)
     (a) Solvable system
     Applying Fredholm's alternative
     $$\underline{y}^T \left(\frac{\partial \underline{R}}{\partial p} \right) = 0$$
     where $\underline{y}$ is a solution of $\underline{\underline{J}}^T \underline{y} = 0$. In this case the problem is solvable but not unique. And we can form
     $$\underline{x}_p = (\underline{x}_p)_{\text{particular}} + \epsilon \underline{z}$$
     where ϵ is a small, arbitrary parameter, $\underline{z}$ is a solution of $\underline{\underline{J}}\,\underline{z} = 0$ and $(\underline{x}_p)_{\text{particular}}$ is a solution of
     $$\underline{\underline{J}} \cdot (\underline{x}_p)_{\text{particular}} = -\underline{R}_p$$
     When $\epsilon = 0$, it gives the slope of the base family. When $\epsilon \neq 0$, it gives the direction of bifurcating family (See Figure 2.2)
     (b) Not solvable
     $$\underline{y}^T \left(\frac{\partial \underline{R}}{\partial p} \right) \neq 0$$
     This case corresponds to the limit point. (See Figure 2.3)
     This is due to the breakdown of representation only, caused by the choice of independent variable.
     One of the trick to avoid this problem is the exchange of independent and dependent variable. That is
     $$\underline{R}(\underline{x}_1, \ldots, \underline{x}_N; p) = 0$$
     is changed to
     $$\underline{R}(\underline{x}_1, \ldots, \underline{x}_{i-1}, p, \underline{x}_{i+1}, \ldots, \underline{x}_N; \underline{x}_i) = 0$$

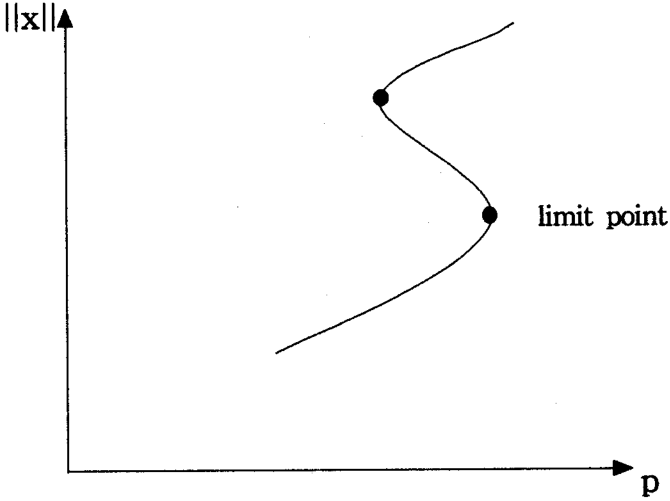

Figure 2.3. Limit point.

Arc-length method :

The other choice is to use arc-length method. In this method, we parametrize $\underline{x}(p)$ as a monotonically increasing function of s. And add the equation

$$N(\underline{x}, p, s) \equiv |s - s_o|^2 - \|\underline{x}(s) - \underline{x}(s_o)\|_2^2 - |p(s) - p(s_o)|^2 = 0$$

New problem is to solve for $\underline{x} = \underline{x}(s)$ and $p = p(s)$ from

$$\begin{aligned} \underline{R}(\underline{x}(s), p(s)) &= \underline{0} \quad \Re^N \\ N(\underline{x}, p, s) &= 0 \quad \Re^1 \end{aligned}$$

Continuation in s

$$\begin{aligned} \underline{x}(s_o + \Delta s) &= \underline{x}(s_o) + \underline{x}_s \Delta s \\ p(s_o + \Delta s) &= p(s_o) + p_s \Delta s \end{aligned}$$

$\underline{x}_s$ and p_s are solved from

$$\underline{\underline{\hat{J}}} \begin{bmatrix} \underline{x}_s \\ p_s \end{bmatrix} = - \begin{bmatrix} \frac{\partial \underline{R}}{\partial s} \\ \frac{\partial N}{\partial s} \end{bmatrix}$$

where

$$\underline{\underline{\hat{J}}} = \begin{bmatrix} \underline{\underline{J}} & \underline{R}_p \\ N_{\underline{x}}^T & N_p \end{bmatrix}$$

and

$$\begin{aligned} J_{ij} &= \frac{\partial R_i}{\partial x_j} \\ (R_p)_i &= \frac{\partial R_i}{\partial p}, \quad i = 1, \ldots, N \\ (N_x)_j &= \frac{\partial N}{\partial x_j}, \quad j = 1, \ldots, N \\ N_p &= \frac{\partial N}{\partial p} \end{aligned}$$

Note :

1. $$\frac{\partial \underline{\underline{R}}}{\partial s} = 0$$
2. At bifurcation point, $\det \underline{\underline{\hat{J}}} = 0$
3. At limit point $\det \underline{\underline{J}} = 0, \det \underline{\underline{\hat{J}}} \neq 0$