The Stability Analysis of ODEs

04/12/04

References:

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1. A nonmathematical introduction to stability analysis

● Bending of a board:

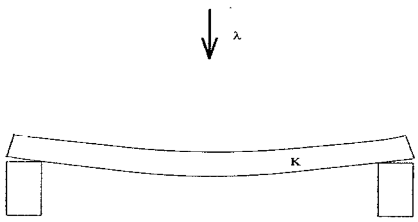


Figure: Bending of a board

-- Quantitative change:

deform within elastic regime(stiffness: K) of board when small λ is applied.

return to original shape of board when the perturbation in λ is removed.

-- Qualitative change (=irreversible change):

abrupt changes when load λ λ_0

-- λ K: control parameters or design parameter

-- Practical research approach: find parameters to control the state of a system

● Some pairs of verbs representing quantitative and qualitative changes

bend	break
incline	tilt over
stretch	tear
inflate	burst
stationary	motion (or oscillatory)
stable	unstable (tiny perturbations may trigger drastic changes)

● Various kinds of qualitative changes

-- Steps in qualitative changes: stationary state, regular motion, irregular motion (regular → irregular: related to turbulence or chaos)

stable ↔ unstable
symmetric ↔ asymmetric
stationary ↔ periodic (regular) motion
regular ↔ irregular
order ↔ chaos

Table: Examples of parameters

Phenomenon	Controlled by a typical parameter
Bending of a board	Load
Vibration of an engine	Frequency of imbalance
Combustion	Temperatures
Nerve impulse	Generating potential
Superheating	Strength of external magnetic field
Oscillation of an airfoil	Speed of plane relative to air
Climatic changes	Solar radiation

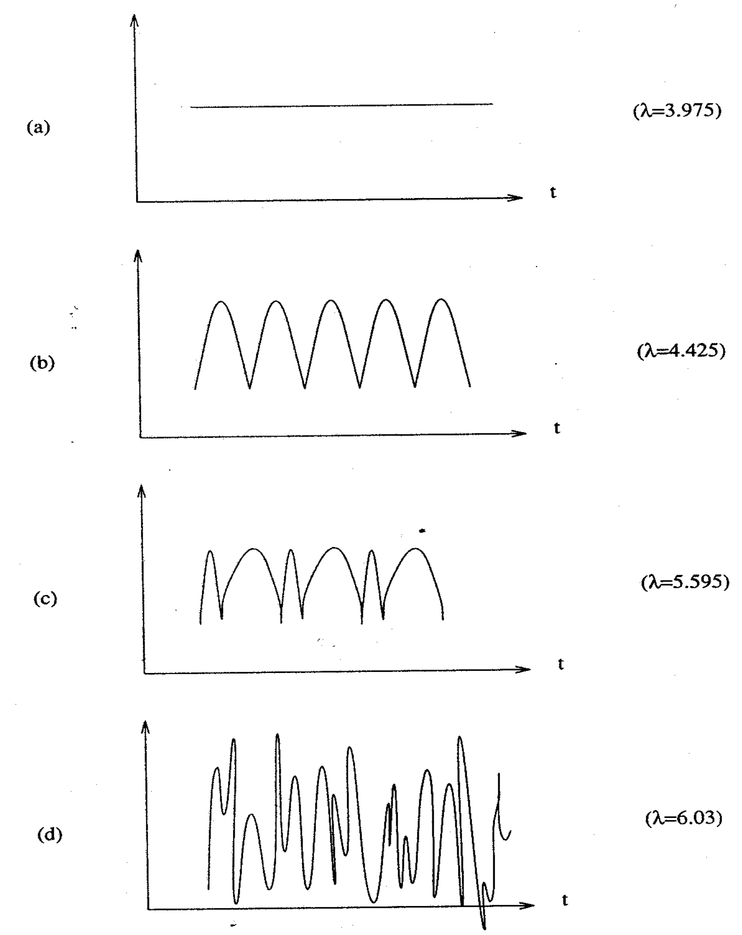


Figure: Velocity of a combustion front (a) stationary (b) wavy (c) wavy, regular (d) irregular (chaotic).

2. Stability analysis of lumped paramter systems (ODEs)

2.1. Lumped parameter systems (ODEs)

$$\dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, \cdots, x_n, t)$$

$$\vdots$$

$$\dot{x}_n = \frac{dx_n}{dt} = f_n(x_1, \cdots, x_n, t)$$

→ vector form: $\dot{\underline{x}} \equiv \frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, t)$

2.2. Definition of stability

Autonomous ODE: $\dot{\underline{x}} = \underline{f}(\underline{x})$

steady state: $\underline{0} = \underline{f}(\underline{x}_s)$

● The stationary solution $\underline{x}_s$ is said to be ***stable*** if the response to a small perturbation remains small as the time approaches infinity. Otherwise the stationary solution is called ***unstable*** (the deviation grows).

(unstable equilibrium is ***source*** and is an example for a ***repellor***)

-- The system is ***stable*** with respect to the region S(x)=0 if $\underline{x}(t)$ remains within the region enclosed by S(x)=0 for all time $0 \leq t < \infty$

A system is considered stable with respect to a region so long as the transient never leaves that region, even though the system may never return to the steady state. ⇒ *Practical stability*

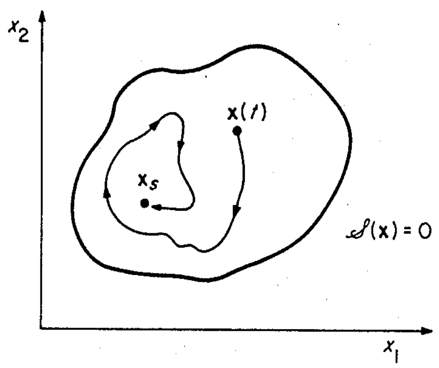


Figure: Movement of a point x(t) in a region of phase space bounded by enclosed surface S(x)=0.

-- A necessary and sufficient condition for stability of this system with respect to the region S(x)=0 is

$\underline{n} \cdot \underline{f}(\underline{x}) \leq 0$ everywhere on $S(\underline{x}) = 0$

● A stationary solution $\underline{x}_s$ is said to be ***asymptotically stable*** if the response to a small perturbation approaches zero as the time approaches infinity
(asymptotically state equilibrium is ***sink*** and is an example for an ***attractor***)

$\underline{x}(t) \rightarrow \underline{x}_s$ at $t \rightarrow \infty$

-- The system is ***asymptotically stable*** with respect to the family of regions $S(x, c) = 0$ if when x(t) lies in a region enclosed by $S(x, c_m)=0$, then $x(t+\Delta$ $\Delta > 0$, lies in a region enclosed by $S(x, c_n) = 0$, $c_n < c_m$.

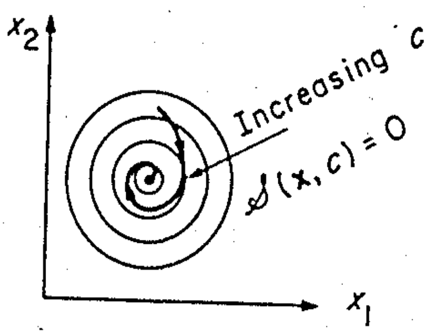


Figure: Asymptotic stability.

-- A necessary and sufficient condition for asymptotic stability of system with respect to the family of regions $S(x, c) = 0$ is

$\underline{n} \cdot \underline{f} < 0$ everywhere on $S(x, c) = 0$, $c < c_\infty$

(c_∞: limiting region of asymptotic stability)

Ex.) Simple case: $\dot{y} = \lambda y$

- solution: $y(t) = \exp(\lambda$, equilibrium: $y_s = 0$
- y_s is stable for $\lambda \leq 0$, asymptotically stable for $\lambda < 0$, and unstable for $\lambda > 0$

-- ***use the term "stable" in the sense "asymptotically stable"***

● Above definitions for stability are ***local*** in nature.
An equilibrium may be stable for a small perturbation but unstable for a large perturbation.

Ex.) $\dot{y} = y(y^2 - \alpha^2)$, $\quad y(0) = z$

- the closed-form solution: $y = 0$ for $z = 0$

$$y(t) = (\alpha^{-2} + (z^{-2} - \alpha^{-2})\exp(2\alpha^2 t))^{-1/2} \text{ for } z \neq$$

- for $|z| > |\alpha$. solutions diverge.
- the domain of attraction ($|z|<|\alpha$) of the stable equilibrium y_s=0 is bounded by two unstable equilibria y_s=±
- although locally stable, y_s=0 is globally unstable when "large" perturbations.

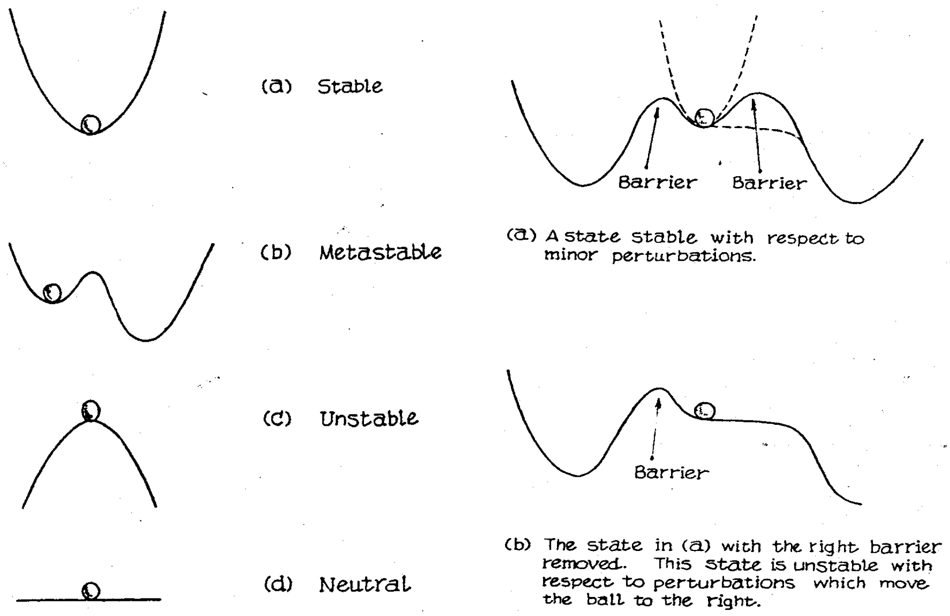


Figure: Stability of a ball.

→ Difficult to obtain the global stable results because systems are so complicated. Importance of linear stability analysis - provide insight into what happens "close" to an equilibrium.

2.3. Linear stability analysis

● Procedure to derive the eigenvalue problems

-- Consider following ODEs

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

-- Steady state solutions: $f_1(x_{1s}, x_{2s}) = f_2(x_{1s}, x_{2s}) = 0$

-- Talyor series expansion of f_1 to linearize above Eq.

$$\dot{x}_1 = f_1(x_{1s}, x_{2s}) + \left(\frac{\partial f_1}{\partial x_1}\right)_s (x_1 - x_{1s}) + \left(\frac{\partial f_1}{\partial x_2}\right)_s (x_2 - x_{2s}) \ + \text{h.o.t.}$$

-- Define deviation variables: $h_1 = x_1 - x_{1s}, \ h_2 = x_2 - x_{2s}$

-- The linearized system in vector notation (linear homogeneous eq.): $\dot{\underline{h}} = \underline{f}^s_x \underline{h}$

where Jacobian matrix, $\underline{f}^s_x = \begin{pmatrix} \frac{\partial f^s_1}{\partial x_1} & \frac{\partial f^s_1}{\partial x_2} \\ \frac{\partial f^s_2}{\partial x_1} & \frac{\partial f^s_2}{\partial x_2} \end{pmatrix}$

-- Insert the *ansatz* (hypothesis), $\underline{h}(t) = e^{\mu t} \underline{w}$

(μ eigenvalues, $\underline{w}$: eigenvectors)

(The only steady state is at $\underline{h}$=0, the solutions will have the above form.)

-- Convert to eigenvalue problem, $(\underline{f}^s_x - \mu \underline{I}) \underline{w} = 0$

-- μ_1, μ_2 are roots of the characteristic eq., $\det(\underline{f}^s_x - \mu \underline{I}) = 0$

※ *The eigenvalues of the Jacobian matrix evaluated at an equilibrium point determine the dynamics behavior in the neighborhood of the equilibrium*

⇒ ***Stability to infinitesimal disturbances (or perturbations)***

⇒ **<u>Liapunov's first method</u>**: Use of the linearized equations to study behavior of a nonlinear system

Ex.) Find all fixed points for $\dot{x} = x^2 - 1$ and classify their stability

Ex.) Determine the stability of the fixed points for $\dot{x} = \sin x$

Ex.) Determine the stability for (a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

Ex.) Duffing eq. (without external forcing)

- consider the 2nd-order ODE

$$\ddot{u} + \dot{u} - u + u^3 = 0$$

- let $y_1 = u$, $y_2 = \dot{u}$

$$\dot{y}_1 = y_2 = f_1(y_1, y_2), \quad \dot{y}_2 = y_1 - y_1^3 - y_2 = f_2(y_1, y_2)$$

- stationary points: (0,0), (1,0), (-1,0)

Jacobian matrix: $\underline{f_y^s} = \begin{pmatrix} 0 & 1 \\ 1 - 3y_1^2 & -1 \end{pmatrix}$

(a) (0,0), characteristic eq.: $0 = \mu^2 + \mu$., root: $\frac{1}{2}(-1 \pm \sqrt{5})$

(b) (± 0), characteristic eq.: $0 = \mu^2 + \mu$ 2, root: $\frac{1}{2}(-1 \pm \sqrt{-7})$

● Types of qualitative behavior of trajectories close to an equilibrium

(a) Nodes: μ_1, μ_2 real, $\mu_1 \cdot$ $_2 > 0$, $\mu_1 \neq$ $_2$.

μ μ > 0: unstable node

(b) Saddle: μ_1, μ_2 real, $\mu_1 \cdot$ $_2 < 0$

always unstable

(c) Foci: μ_1, μ_2 complex conjugate with nonzero real part, $\mu_1 = \alpha$ β $\mu_2 = \alpha$ β

α α > 0: unstable focus

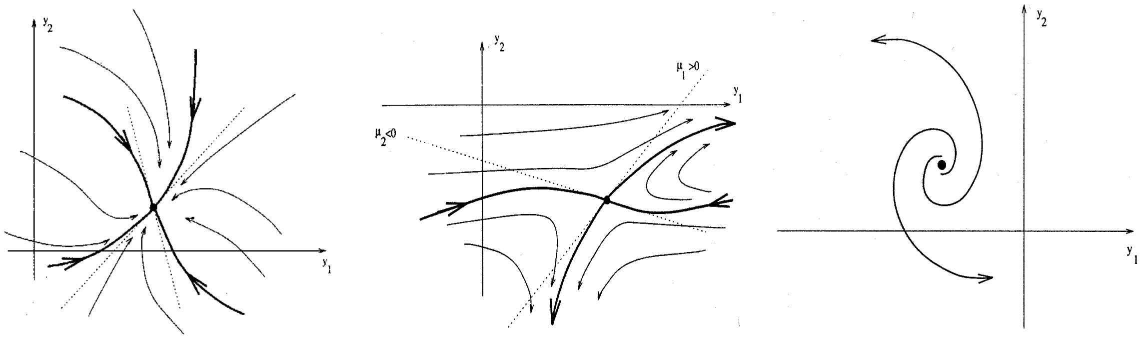


Figure: Stable node *Saddle* *Focus*

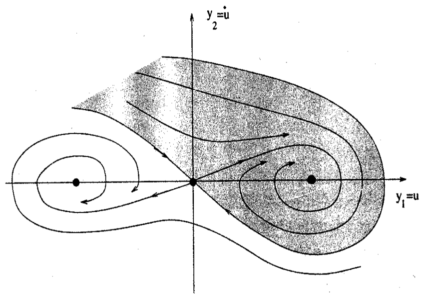


Figure: Phase plane of the Duffing equation.

- Degenerate cases; parameter dependence

  $\mu_1=\mu_2$ (a special node)

  $\mu_1 \cdot \ _2=0$ (require nonlinear terms)

  $\mu_{1,2} = \pm$? (a center of concentric cycles)

Definition: The equilibrium is called ***hyperbolic*** or ***nondegenerate*** when the Jacobian $\underline{f}_{\underline{x}}^{s}$ has no eigenvalue with zero real part. The exceptional cases $\mu_1 \cdot \ _2=0$ and $\mu_{1,2} = \pm$? are called ***nonhyperbolic*** or ***degenerate***.

-- Consider $\dot{\underline{x}} = \underline{f}(\underline{x}, \lambda)$,

Upon varying the parameter λ the position and qualitative features of a stationary point can vary. In other words, qualitative changes such as a loss of stability are encountered when a degenerate case is passed.

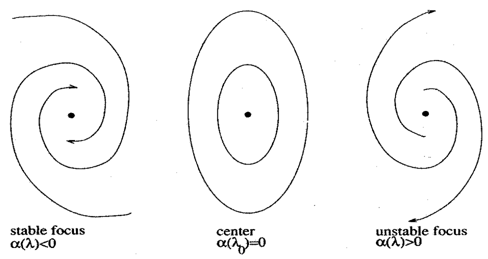


Figure: A focus changes its stability for λ λ_0.

- Generalizations: the principle of linear stability

Theorem: Suppose $\underline{f}(\underline{x})$ is two continuously differentiable and $\underline{f}(\underline{x}_s)=0$. The real parts of the eigenvalues μ_j (j=1,⋯ ı) of the Jacobian evaluated at the stationary solution $\underline{x}_s$ determine stability in the following way;

(a) Re (μ_j) < 0 for all j implies asymptotic stability

(b) Re (μ_j) > 0 for one (or more) k implies instability

2.4. Applications of linear stability analysis

-- Remind the previous results for stability analysis

Introduction

(a) 2nd-order polynomials: $\lambda^2 + a_1\lambda + a_2 = 0$ (root: λ_1, λ_2)

- For the roots of above equation to have negative real parts, it is necessary and sufficient that a_1 and a_2 be positive.

(b1) Higher order polynomials: $a_o\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

- use the Routh-Hurwitz criterion
- form the n determinants

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_o & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ 0 & a_0 & a_2 & \cdots & 0 \\ 0 & 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & & & a_i \end{vmatrix} \quad (i = 1, \cdots n), \quad a_k = 0,\ k > n$$

- the n roots of above equation will have negative real parts if and only if $\Delta_i > 0$, $i = 1, 2, \cdots n$.

(b2) Routh stability criterion - determining stability without calculating eigenvalues

$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

- Routh array

1	a_n	a_{n-2}	a_{n-4}
2	a_{n-1}	a_{n-3}	a_{n-5}
3	b_1	b_2	b_3
4	c_1	c_2	c_3
n+1			

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \quad b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1}b_2}{b_1}, \quad c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

- A necessary and sufficient condition for all roots of the characteristic polynomial to have negative real part is that all of the coefficients of the polynomial are positive and all of the elements in the left column of Routh array are positive.

Application 1: CFSTR (continuous flow stirred tank reactor)

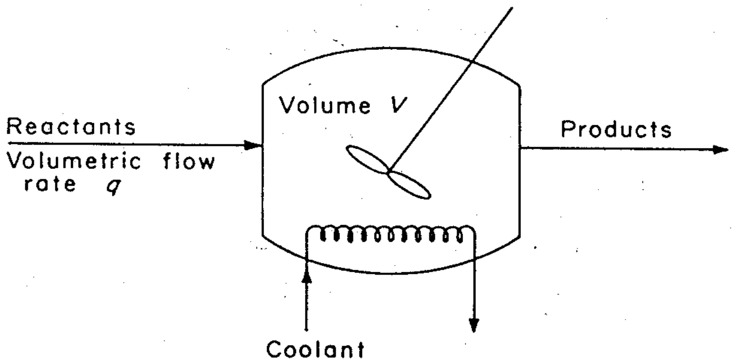


Figure: Schematic diagram of a CFSTR.

● Governing equations (mass & energy balance)

-- Assume the single liquid phase chemical reaction, A → products

$$V\frac{dc}{d\tilde{t}} = q[c_f - c] - Vr(c, T)$$

$$\rho c_p V\frac{dT}{d\tilde{t}} = \rho c_p q[T_f - T] + [-\Delta H]Vr(c, T) - Ua[T - T_{cf}]$$

$$r(c, T) = kce^{-E/RT}$$

Notations:

$\tilde{t}$: time, V: volume, q: volumetric flow rate, c: conc. of reactant A, c_f: feed conc., ρ density, c_p: heat capacity, T: temp., T_f: feed temp., T_{cf}: coolant feed temp., Δ [: heat of reaction, U: overall heat transfer coeff., a: area available for heat transfer, K: constant, E: activation energy, R: gas constant

● Dimensionless governing equations

-- Dimensionless variables:

$$x = \frac{c}{c_f}, \quad y = \frac{T}{T_f}, \quad \alpha = \frac{kV}{q}, \quad \delta = \frac{Ua}{\rho q c_p}$$

$$\beta = \frac{-\Delta H c_f}{\rho c_p T_f[1+\delta]}, \quad \gamma = \frac{E}{RT_f}, \quad t = \frac{q\tilde{t}}{V}, \quad \phi = \frac{1+\delta T_{cf}/T_f}{1+\delta}$$

(β s positive for an exothermic reaction)

$$\frac{dx}{dt} = 1 - x - \alpha x e^{-\gamma/y}$$

$$\frac{1}{1+\delta}\frac{dy}{dt} = \phi - y + \alpha\beta x e^{-\gamma/y}$$

(for an adiabatic reactor, heat transfer area goes to zero, $\delta \to 0$, $\phi \to 1$)

● **Steady state CFSTR**

$$0 = 1 - x_s - \alpha x_s e^{-\gamma/y_s}$$

$$0 = \phi - y_s + \alpha\beta x_s e^{-\gamma/y_s}$$

$\rightarrow \quad \beta x_s = \beta + \phi - y_s \quad (0 \le x_s \le 1,\ \phi \le y_s \le \phi\ \beta$

$\rightarrow \quad \frac{1}{\alpha}[y_s - \phi] = F(y_s), \quad F(y) = [\beta + \phi - y]e^{-\gamma/y}$

This system have multiple steady state solutions (result of uniqueness condition)

● **Stability to infinitesimal perturbations: CFSTR**

-- Stability of CFSTR

$$\frac{dx}{dt} = 1 - x - \alpha x e^{-\gamma/y} = f_1(x,y)$$

$$\frac{dy}{dt} = [1+\delta](\phi - y + \alpha\beta x e^{-\gamma/y}) = f_2(x,y)$$

-- Components of Jacobian matrix:

$$a_{11} = -1 - \alpha e^{-\gamma/y_s}, \quad a_{12} = -\frac{\alpha\gamma}{\beta y_s^2}F(y_s)$$

$$a_{21} = [1+\delta]\alpha\beta e^{-\gamma/y_s}, \quad a_{22} = -[1+\delta] + \frac{[1+\delta]\alpha\gamma}{y_s^2}F(y_s)$$

characteristic eq.: $\lambda^2 - [a_{11} + a_{22}]\lambda + [a_{11}a_{22} - a_{12}a_{21}] = 0$

-- For roots with negative real parts,

$$a_{11}a_{22} - a_{12}a_{21} = [1+\delta][1 - \alpha F'(y_s)] > 0$$

$$a_{11} + a_{22} = -2 + \alpha F'(y_s) - \delta\left[1 - \frac{\alpha\gamma}{y_s^2}F(y_s)\right] < 0$$

$\rightarrow \quad \alpha F'(y_s) < 1$

$$\alpha F'(y_s) < 2 + \delta\left[1 - \frac{\alpha\gamma}{y_s^2}F(y_s)\right]$$

- necessary and sufficient conditions for a steady state to be stable to infinitesimal perturbations

-- Marginal stability (=neutral stability)

$$\lambda_I = 0: \ \alpha_c = \frac{1}{F'(y_s)}$$

$$\lambda_I \neq 0: \ \alpha_c = \frac{2 + \delta[1 - (\alpha\gamma/y_s^2)F(y_s)}{F'(y_s)}$$

Application 2: Anisotropic fluid

- **Incompressible Newtonian fluid:** $\tau_{ij} + p\,\delta_{ij} = 2\mu d_{ij}$ (i,j=1,2,3)

  where $d_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right]$, $\delta_{ij} = 1_{\{i=j\}}$ or $0_{\{i \neq j\}}$

  p=isotropic pressure, $\underline{v}$=velocity vector, $\underline{x}$=coordinate location,
  $\boldsymbol{\tau}$=extra-stress tensor, μ viscosity

- **Simple shear flow**

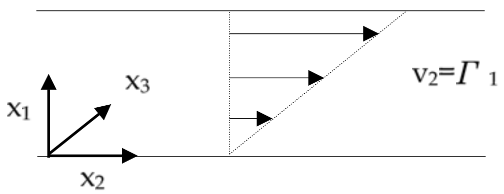


Figure: Simple shear flow

  $v_1=v_3=0$, $v_2=\Gamma\,{}_1$ (Γ shear rate)

  the only nonzero stresses are the shear stresses: $\tau_{12} = \tau_{21} = \mu\Gamma$

- Polymer solutions, polymer melts, fiber suspensions, and liquid crystals, because of the internal structure in the fluid, do not follow this simple relation. A class of structural theories has been developed which relate the stress to the local structure.

-- In the simplest of these theories (developed by J. L. Ericksen), the structure in the liquid is described by the magnitude and orientation of a vector, **n**.

-- $\tau_{ij} + p\,\delta_{ij} = \left[\mu_1 \sum_{r,s=1}^{3} d_{rs} n_r n_s\right] n_i n_j + 2\mu_2 d_{ij} + 2\mu_3 \sum_{k=1}^{3} [d_{ik} n_k n_j + n_i d_{jk} n_k]$

$$\frac{\partial n_i}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial n_i}{\partial x_k} - \frac{1}{2} \sum_{k=1}^{3} \left[\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right] n_k$$

$$= \left[\beta_1 + \beta_2 \sum_{r,s=1}^{3} d_{rs} n_r n_s\right] n_i + \beta_3 \sum_{k=1}^{3} d_{ik} n_k$$

-- In simple shear flow, shear stress & orientation vector equations simplify to

$$\tau_{12} = \left(\mu_1 n_1^2 n_2^2 + \mu_2 + \mu_3[n_1^2 + n_2^2]\right)\Gamma$$

$$\frac{dn_1}{dt} = [\beta_1 + \beta_2 \Gamma n_1 n_2] n_1 + \frac{1}{2}[\beta_3 - 1]\Gamma n_2$$

$$\frac{dn_2}{dt} = [\beta_1 + \beta_2 \Gamma n_1 n_2] n_2 + \frac{1}{2}[\beta_3 + 1]\Gamma n_1$$

$$\frac{dn_3}{dt} = [\beta_1 + \beta_2 \Gamma n_1 n_2] n_3$$

  where β_1, β_2, β_3, μ_1, μ_2, μ_3 are constants and n is assumed to independent spatially.

- **The possibility of multiple solutions:** see next copies

Application 3: Feedback control

- $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{b}(\underline{x})u$

-- u: a scalar control variable which regulate performance in the neighborhood of the steady state and vanishes for $\underline{x} = \underline{x}_s$. ($u = \underline{k} \cdot [\underline{x} - \underline{x}_s]$)

→ $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{b}(\underline{x})\underline{k} \cdot [\underline{x} - \underline{x}_s]$

-- For small perturbations, following equation with deviation variables is written

$$\dot{\underline{h}} = [\underline{\nabla} f(\underline{x}_s) + \underline{b}\,\underline{k}] \cdot \underline{h} + O(\underline{h})$$

Ex.) $x''' = g(x, x', x'') + bu$ ($u = k x_1$: proportional to the offset in x)

$$\dot{x}_1 = x_2,\quad \dot{x}_2 = x_3,\quad \dot{x}_3 = g(\underline{x}) + bkx_1$$

- Jacobian matrix = $\underline{\nabla} f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- The eigenvalue eq. for A+bk is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a_{31} + bk & a_{32} & a_{33} - \lambda \end{vmatrix} = -\lambda^3 + a_{33}\lambda^2 + a_{32}\lambda + [a_{31} + bk] = 0$$

- For eigenvalues with negative real parts,

$a_{33} < 0$, $a_{32} < 0$, $a_{31} + bk < 0$, $a_{33} a_{32} + a_{31} > -bk$

- Marginal stability: $-bk_c = a_{33} a_{32} + a_{31}$

- Remarks about linear stability analysis

(a) No information is obtained about how large a perturbation can be tolerated before instability will occur.

(b) Only information about asymptotic stability is obtained

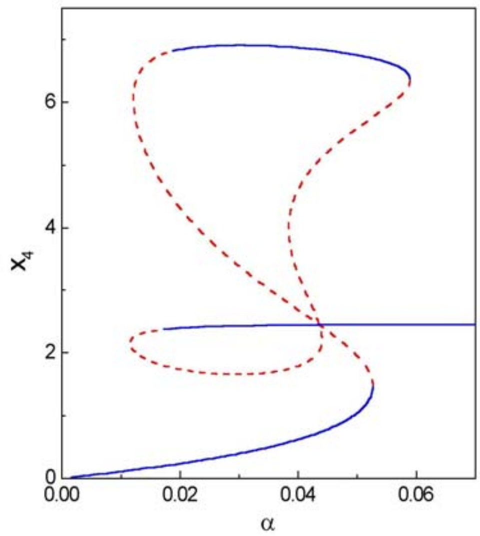
Application 4: Dynamical behavior of a cascade of two CSTR with recylcle (1st order exothermic reaction)

$$\frac{dx_1}{dt} = (1-\Lambda)x_2 - x_1 + \alpha(1-x_1)\exp\left(\frac{x_3}{1+x_3/\gamma}\right) = 0$$

$$\frac{dx_2}{dt} = x_1 - x_2 + \alpha(1-x_2)\exp\left(\frac{x_4}{1+x_4/\gamma}\right) = 0$$

$$\frac{dx_3}{dt} = (1-\Lambda)x_4 - x_3 + \alpha B(1-x_1)\exp\left(\frac{x_3}{1+x_3/\gamma}\right) - \beta_1(x_3-\theta_{c1}) = 0$$

$$\frac{dx_4}{dt} = x_3 - x_4 + \alpha B(1-x_2)\exp\left(\frac{x_4}{1+x_4/\gamma}\right) - \beta_2(x_4-\theta_{c2}) = 0$$



Dependence of the steady-state solution on the parameter

Parameter: $\gamma = 1000,\ B = 22, \beta_1 = \beta_2 = 2,\ \Lambda = 1, \theta_{c1} = \theta_{c2} = 0$

(Blue line: stable steady state, dotted red line: unstable steady state)