Nonlinear Systems Analysis
II. Case Study of the Quadratic Map

Objectives:

- See the similarity between discrete time dynamic models and numerical methods
- Determine the asymptotic stability of a solution to the quadratic map
- Understand the concept of a bifurcation

Bifurcation parameter: the parameter changing the number and character of solutions of the system.
(Asymptotically stable → periodic solution → chaos…)

Chaos: exhibits sensitivity to initial conditions, impossible long-term predictability.
may occur in a single nonlinear discrete equation
(population growth model in this chapter)

(*3 continuous ODEs are required in continuous models. i.e., Lorenz model*)

<u>Noninear dynamics, dynamical systems theory, nonlinear science:</u>
terms for the branch of mathematics related to chaos.

1. A Simple Population Growth Model

$$n_{k+1} = n_k + b_k - d_k$$

(n_k, n_{k+1}: population at the time period k and k+1
b_k, d_k: number of births and deaths during k)

Simple relationship: $n_{k+1} = n_k + \alpha_b n_k - \alpha_d n_k = (1+\alpha_b - \alpha_d) n_k = \alpha n_k$

(similar to fixed-point iteration) $\Rightarrow n_{k+1} = \alpha^k n_0$

$\alpha<1$: population decreases
$\alpha>1$: increases
$\alpha=1$: remains

Unrealistic…
more refined model required

2. Quadratic Map (Logistic Equation)

$x_{k+1} = \alpha x_k (1 - x_k) = g(x_k)$: discrete dynamic equation

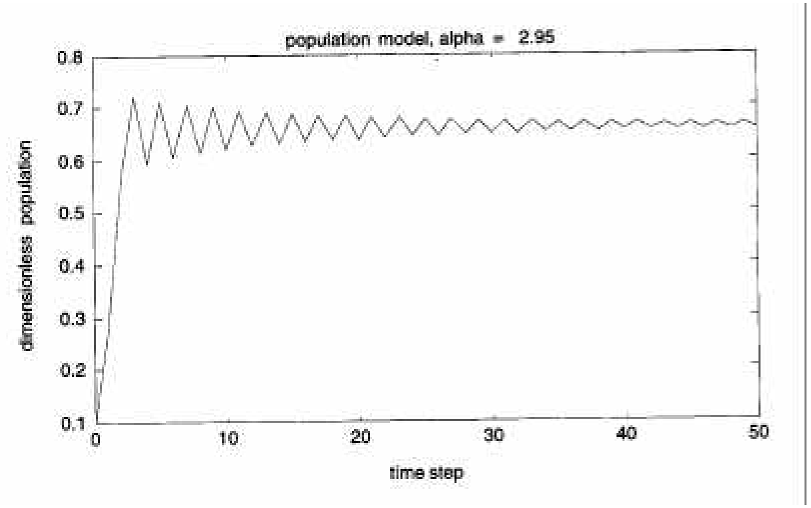
Steady state solutions: $x_s = 0 \text{ and } \dfrac{\alpha - 1}{\alpha}$

- Predicted non-zero solutions for four cases:

$\alpha = 2.95,\ x_s = 0.6610;\quad \alpha = 3.20,\ x_s = 0.6875$

$\alpha = 3.50,\ x_s = 0.7143;\quad \alpha = 3.75,\ x_s = 0.7333$

- Transient response results for four cases:



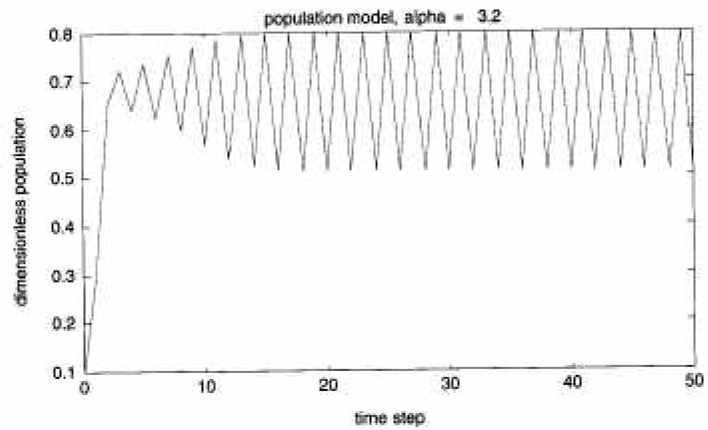
(a) a=2.95:

Asymptotically stable behavior

(b) a=3.20:

Periodic behavior (2-period)

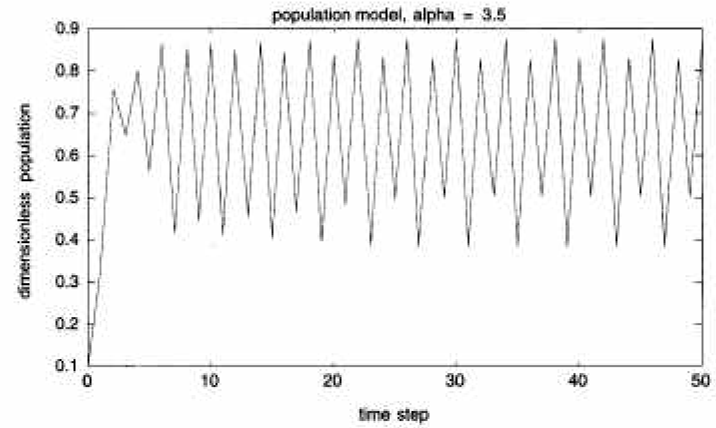
(oscillation between 0.513 and 0.800)



(c) a=3.50:

Periodic behavior (4-period)

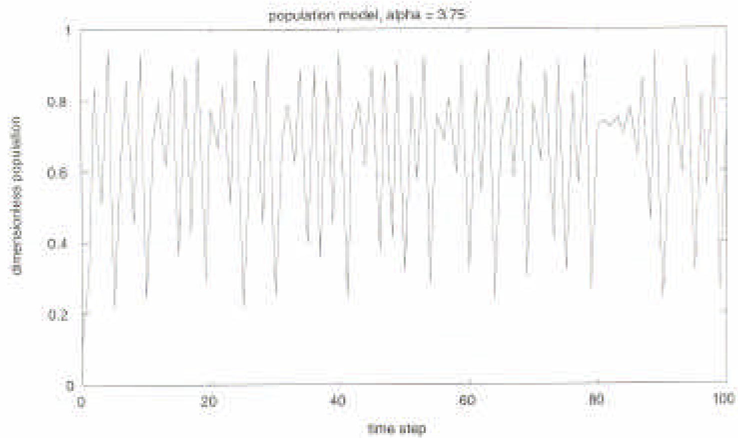
(oscillation between 0.383, 0.827, 0.501, and 0.875)



(d) a=3.75:

Chaotic behavior

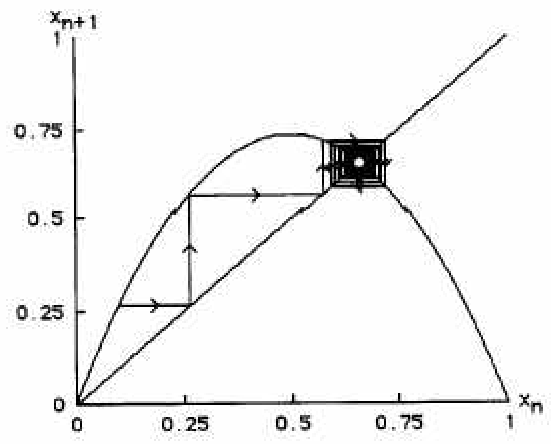
Highly sensitive to initial conditions



3. Cobweb Diagrams

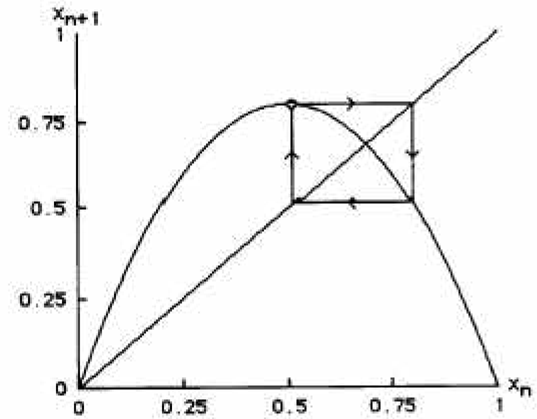
(a) a=2.95:

Converge to the steady-state (x_s=0.6610)



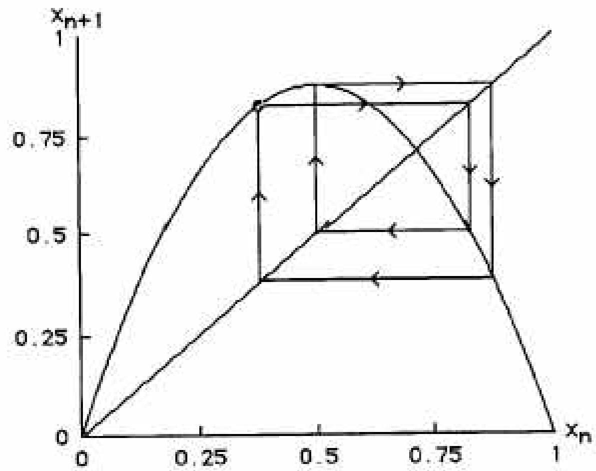
(b) a=3.20:

(2-period)



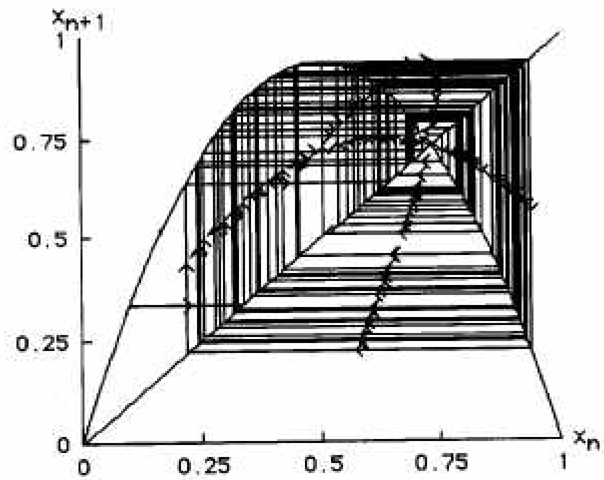
(c) a=3.50:

(4-period)

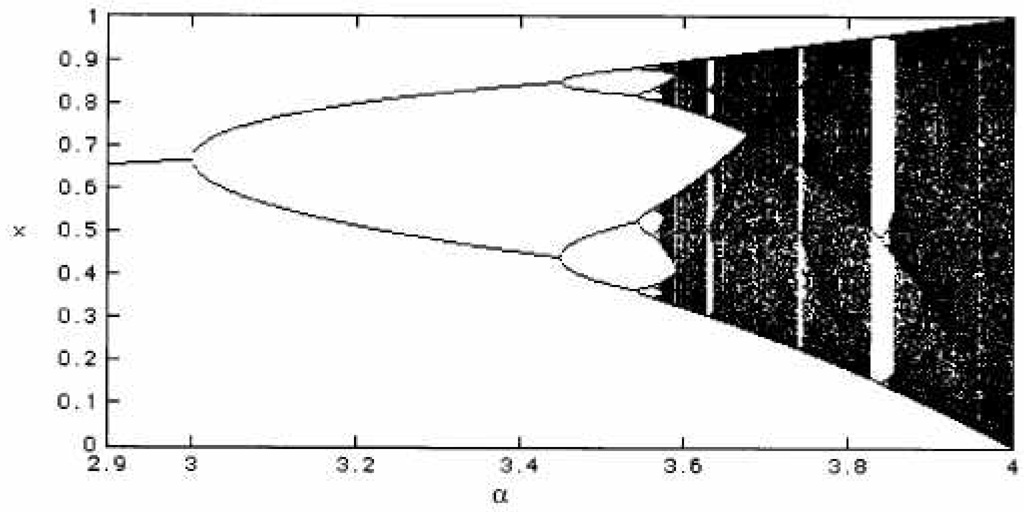


(d) a=3.75:

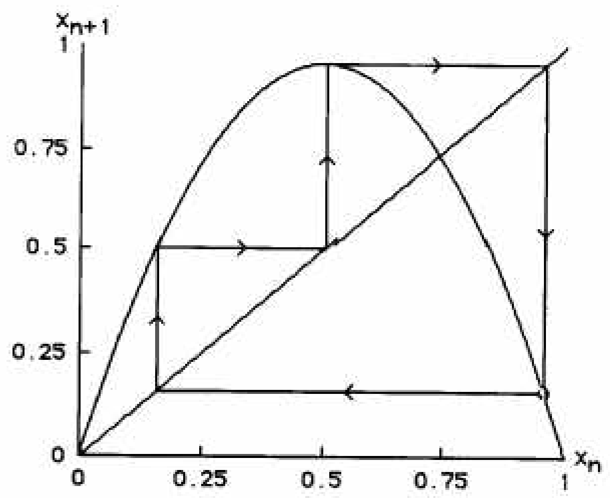
(chaos)



4. Bifurcation and Orbit Diagrams



- Single steady-state (α<3)
- Bifurcation to two solutions occurs at α=3 (period-2)
- New bifurcation occurs at α=3.44949 (period-4), period-8 at α=3.54409, period-16 at α=3.564407, period-32 at α=3.568759, period-64 at α=3.569692, chaos at α=3.56995…
- Period-3 behavior at α=3.83



5. Stability of Steady-State Solutions

- **Definition:** Let x* represent the fixed-point solution of x*=g(x*) or g(x*)-x*=0

Theorem: x* is a stable solution of x*=g(x*), if $\left|\frac{\partial g}{\partial x}\right| < 1$ when evaluated at x*.

(*For the derivation of this theorem, see the previous notes explaining the fixed point iteration scheme.*)

- Stability results for four cases

Case	α	x*	Ig'(x*)I	condition	x*	Ig'(x*)I	condition
1	2.95	0	2.95	Unstable	0.6610	0.9499	Stable
2	3.20	0	3.20	Unstable	0.6875	1.2000	Unstable
3	3.50	0	3.50	Unstable	0.7143	1.5000	Unstable
4	3.75	0	3.75	Unstable	0.7333	1.7500	Unstable

- Stability of x_0^* (x_s=0) as a function of a

$$\left|g'(x_0^*)\right| = \left|\alpha\right|$$

→ x_0^* is stable for α<1 (no physical meaning for negative α)

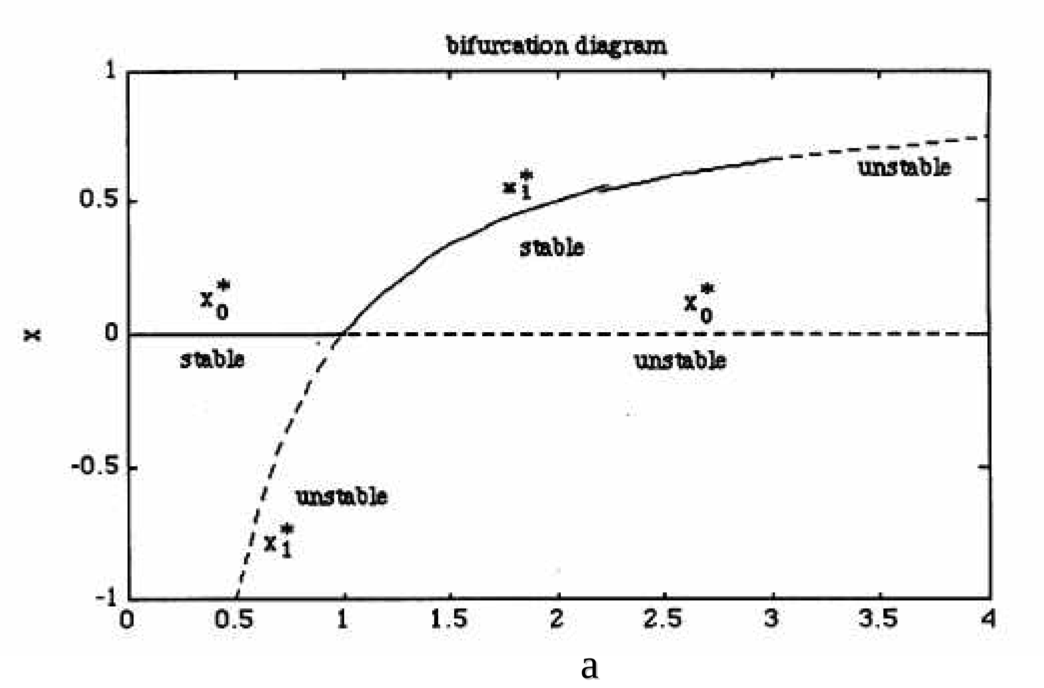
- Stability of x_1^* (x_s= (a-1)/ a) as a function of a

$$\left|g'(x_0^*)\right| = \left|-\alpha + 2\right|$$

→ x_1^* is stable for 1<α<3

- Bifurcation diagram

(Transcritical bifurcation
in next chapter)



- Response of the system

g'	Stability	x*
< -1	unstable	oscillatory
-1 < g' < 0	stable	oscillatory
0 < g' < 1	stable	monotonic
>1	unstable	monotonic

- Feigenbaum's number: $$\lim_{i\to\infty}\frac{\alpha_i-\alpha_{i-1}}{\alpha_{i+1}-\alpha_i}=4.669196223$$