5. Macroscopic Balances

5.1 Introduction

Conservation of mass, energy, and momentum

⇒ Mathematical models of the flow

: A quantitative description of a physical flow process

: A set of mathematical relations

Macroscopic models, in which we are interested only in overall process performance, and not in the detailed structure of the flow field.

5.2 Control Volume and Conservation Principle

Control volume:

a region of space with well-defined boundaries where we can monitor the flow in and out of the quantity that is being conserved

Conservation principle:

The rate of change of the conserved quantity within the control volume	=	The rate at which the conserved quantity enters the control volume	−	The rate at which the conserved quantity leaves the control volume

Fig. 5-1. One-dimensional flow.

Fig. 5-2. Differential area with velocity vector **v** and norma component V.

The differential volumetric flow rate through the small surface element dA is

$$d(\text{volumetric flow rate}) = \mathbf{v} \cdot d\boldsymbol{A} = V dA$$

The differential mass flow rate . . .

$$d(\text{mass flow rate}) = \rho\, \mathbf{v} \cdot d\boldsymbol{A} = \rho V dA$$

The differential flow rate of CQ (the conserved quantity) . . .

$$d(\text{flow rate of } CQ) = \rho(cq)\, \mathbf{v} \cdot d\boldsymbol{A} = \rho(cq) V dA$$

(cq : the amount per unit mass)

The total flow rate of CQ over the surface is

$$\int_{surface} \rho(cq) V dA = \langle \rho(cq) V \rangle A$$

The surface average of Ψ, a quantity that varies from position to position on the surface:

$$<\Psi> = \frac{1}{A}\int_{surface} \Psi \, dA$$

The total amount of CQ in the control volume:

$$CQ \text{ in control volume} = \int_{z_1}^{z_2}\int_{surface} \rho(cq)\, dA\, dz$$

$$= \int_{z_1}^{z_2} <\rho(cq)> A\, dz$$

Example 5.1 $\Psi = \Psi_m\left(1 - \frac{r^2}{R^2}\right)$, $\langle \Psi \rangle = ?$

Fig. 5-3. Differential area in polar coordinates.

$$\langle \Psi \rangle = \frac{1}{\pi R^2} \int_{r=0}^{R} \int_{\theta=0}^{2\pi} \Psi_m \left(1 - \frac{r^2}{R^2}\right) r\, d\theta\, dr$$

$$= \frac{2\pi \Psi_m}{\pi R^2} \int_0^R \left(1 - \frac{r^2}{R^2}\right) r\, dr$$

$$= 2\Psi_m \int_0^1 (1 - \xi^2)\xi\, d\xi$$

$$= \frac{1}{2} \Psi_m$$

Example 5.2

Fig. 5-4. Rectangular cross section with different values of Ψ in the two parts.

$$
\begin{aligned}
\langle \Psi \rangle &= \frac{1}{WH}\int_{y=0}^{y=H} \Psi W\, dy = \frac{W}{WH}\left[\int_{y=0}^{\lambda H} \Psi_1\, dy + \int_{\lambda H}^{H} \Psi_2\, dy\right] \\
&= \frac{W}{WH}\left[\Psi_1 \lambda H + \Psi_2 (H - \lambda H)\right] = \lambda \Psi_1 + (1-\lambda)\Psi_2
\end{aligned}
$$

5.3 Conservation of Mass

Basic equation: (continuity equation)

$$\frac{d}{dt}\int_{z_1}^{z_2} <\rho> A\,dz = <\rho V>_1 A_1 - <\rho V>_2 A_2$$

or letting $w = <\rho V> A$,

$$\frac{d}{dt}\int_{z_1}^{z_2} <\rho> A\,dz = w_1 - w_2$$

At steady state, $w_1 = w_2 = w$

Single fluid:

In a single fluid phase, the density does not change over a cross section of the conduit.

$$\int_{area} \rho (cq) V dA \approx \rho \int_{area} (cq) V dA$$

or $\quad \langle \rho (cq) V \rangle \approx \rho \langle (cq) V \rangle$

The continuity equation becomes

$$\frac{d}{dt} \int_{z_1}^{z_2} \langle \rho \rangle A dz = \rho_1 \langle V \rangle_1 A_1 - \rho_2 \langle V \rangle_2 A_2$$

If the fluid is incompressible,

$\rho_1 = \rho_2 = \rho \qquad$ and the volume is a constant

$$\therefore \ \langle V \rangle_1 A_1 = \langle V \rangle_2 A_2 \quad \text{or} \quad \frac{\langle V \rangle_1}{\langle V \rangle_2} = \frac{A_2}{A_1}$$

5.4 Conservation of Energy

Basic equation:

The first law of thermodynamics for a flowing system:

The rate of change of the total energy within the control volume	=	the rate at which energy enters the control volume by flow	−	the rate at which energy leaves the control volume by flow
	+	the rate at which heat is added through the boundaries	+	the rate at which the work is done on the fluid in the control volume

The total energy = the internal energy + the potential energy
+ the kinetic energy

or $cq = e + \frac{1}{2}\upsilon^2 + gh$

Work = flow work + shaft work

Flow work : the work required to move the fluid into and out of the control volume = $\langle pV \rangle_1 A_1 - \langle pV \rangle_2 A_2$

Shaft work : all other work done on the fluid, $\dot{W}_S$

Then, the equation of conservation of energy is

$$\frac{d}{dt}\int_{z_1}^{z_2} \langle \rho(e+\frac{1}{2}\upsilon^2+gh)\rangle A dz =$$

$$\langle \rho(e+\frac{1}{2}\upsilon^2+gh)V\rangle_1 A_1 - \langle \rho(e+\frac{1}{2}\upsilon^2+gh)V\rangle_2 A_2$$

$$+\langle pV\rangle_1 A_1 - \langle pV\rangle_2 A_2 + \dot{Q}_H + \dot{W}_S$$

Simplifying assumptions:

Assumption 1: Steady state, $d/dt = 0$

Assumption 2: Single phase, uniform properties

$$<\rho v^2 V> \approx \rho < v^2 V, \qquad <\rho e V> \approx \rho e < V >$$

Assumption 3: Uniform equivalent pressure

$$<(p+\rho g h)V> \approx (p+\rho g h)<V>$$

Then, the energy balance becomes

$$e_1(\rho_1 < V >_1 A_1) + \frac{1}{2}\frac{< v^2 V >_1}{< V >_1}(\rho_1 < V >_1 A_1) + g h_1(\rho_1 < V >_1 A_1)$$

$$- e_2(\rho_2 < V >_2 A_2) - \frac{1}{2}\frac{< v^2 V >_2}{< V >_2}(\rho_2 < V >_2 A_2) - g h_2(\rho_2 < V >_2 A_2)$$

$$+ \frac{p_1}{\rho_1}(\rho_1 < V >_1 A_1) - \frac{p_2}{\rho_2}(\rho_2 < V >_2 A_2) + \dot{Q}_H + \dot{W}_S = 0$$

or

$$e_1 + \frac{1}{2}\frac{<\upsilon^2 V>_1}{<V>_1} + gh_1 + \frac{p_1}{\rho_1}$$

$$-e_2 - \frac{1}{2}\frac{<\upsilon^2 V>_2}{<V>_2} - gh_2 - \frac{p_2}{\rho_2} = -\frac{\dot{Q}_H}{w} - \frac{\dot{W}_S}{w}$$

defining $\delta Q_H = \frac{\dot{Q}_H}{w}$, $\delta W_S = \frac{\dot{W}_S}{w}$, it finally becomes

$$\Delta\left(e + \frac{1}{2}\frac{<\upsilon^2 V>}{<V>} + gh + \frac{p}{\rho}\right) = \delta Q_H + \delta W_S$$

Example 5.3

Compute the temperature rise for adiabatic flow of a nonreacting incompressible fluid in a horizontal pipe of uniform cross section, and the heat removal required to keep the flow isothermal.

The pipe is horizontal $\Rightarrow$ $\Delta h = 0$

The cross section is uniform $\Rightarrow$ $\Delta(<v^2 V>/<V>) = 0$

The nonreacting incompressible fluid $\Rightarrow$ $\Delta e = c_v \Delta T$

Assume no shaft work

Then, $c_v \Delta T + \dfrac{\Delta p}{\rho} = \delta Q_H$

If the flow is adiabatic, $\delta Q_H = 0$, and $\Delta T = \dfrac{-\Delta p}{\rho c_v}$

If isothermal, $\Delta T = 0$, and $\delta Q_H = \dfrac{\Delta p}{\rho}$

Velocity averages:

Let's introduce $\alpha = \dfrac{<\upsilon^2 V>}{<V>^3}$

Then, we have simpler form

$$\Delta\left(e + \frac{\alpha}{2}<V>^2 + gh + \frac{p}{\rho}\right) = \delta Q_H + \delta W_S$$

If $\upsilon = V$, $\alpha = \dfrac{<V^3>}{<V>^3}$

For turbulent pipe flow : $\alpha \approx 1.07$

For laminar flow : $\alpha = 2.0$

Engineering Bernoulli Equation:

Differential form of energy balance

$$de + \frac{1}{2} d(\alpha \langle V \rangle^2) + g\,dh + d\left(\frac{p}{\rho}\right) = dQ_H + dW_S$$

$$de = T\,ds - p\,d\left(\frac{1}{\rho}\right)$$

$$= T\,ds - d\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\,dp$$

Then, we have

$$(T\,ds - dQ_H) + \frac{1}{2} d(\alpha \langle V \rangle^2) + g\,dh + \frac{dp}{\rho} = dW_S$$

From the second law of thermodynamics,

$T\,ds - dQ_H \equiv dl_V \geq 0$ (zero only for a reversible process)

$$\therefore \frac{1}{2} d(\alpha < V >^2) + g\,dh + \frac{dp}{\rho} = dW_S - dl_V$$

Integrating from z_1 to z_2, we obtain

$$\frac{\alpha_2}{2} < V >_2^2 + gh_2 = \frac{\alpha_1}{2} < V >_1^2 + gh_1 - \int_{p_1}^{p_2} \frac{dp}{\rho} + \delta W_S - l_V$$

: Engineering Bernoulli Equation

: Mechanical Energy Balance

Isothermal, ideal gas:

$$\int_{p_1}^{p_2} \frac{dp}{\rho} = \frac{R_g T}{M_w} \int_{p_1}^{p_2} \frac{dp}{p} = \frac{R_g T}{M_w} \ln \frac{p_2}{p_1}$$

The fluid is incompressible:

$$\int_{p_1}^{p_2} \frac{dp}{\rho} = \frac{1}{\rho} \int_{p_1}^{p_2} dp = \frac{p_2 - p_1}{\rho}$$

Equivalent heads:

$$\frac{\alpha_2}{2g} < V >_2^2 + h_2 + \frac{p_2}{\rho_2 g} = \frac{\alpha_1}{2g} < V >_1^2 + h_1 + \frac{p_1}{\rho_1 g} + \frac{\delta W_S}{g} - \frac{l_V}{g}$$

$\frac{1}{2} < V >^2$: velocity head

* 1 velocity head ≈

the losses in turbulent flow in a pipe 50 diameters long

Pipeline losses:

Losses in the straight lengths of pipe + losses in the fittings

Losses in the straight pipe : $l_V = \frac{p_1 - p_2}{\rho} = \frac{2 < V >^2 Lf}{D}$

Losses in fittings : $l_V = \frac{1}{2} <V>^2 K_f$

In a pipeline network,

$$l_V = \sum_{\text{pipe lengths}} \frac{2<V>_i^2 L_i f_i}{D_i} + \sum_{\text{fittings}} \frac{1}{2} <V>_i^2 K_{fi}$$

Table 5-1. K_f in fittings and valves for turbulent flow

5.5 Conservation of Linear Momentum

Basic equation:

The rate of change of the linear momentum within the control volume	=	the rate at which linear momentum enters the control volume by flow	−	the rate at which linear momentum leaves the control volume by flow
	+	the sum of all forces acting on the system		

Linear momentum is a vector quantity.

Linear momentum per unit mass is simply the velocity vector, $\boldsymbol{v}$. Then, the conservation of momentum for a single fluid phase is

$$\frac{d}{dt}\int_{z_1}^{z_2}\rho<\boldsymbol{v}>A\,dz = \rho_1<\boldsymbol{v}V>_1A_1-\rho_2<\boldsymbol{v}V>_2A_2$$
$$+p_1\boldsymbol{A_1}-p_2\boldsymbol{A_2}-\boldsymbol{F}+(\int_{z_1}^{z_2}\rho A\,dz)\boldsymbol{g}$$

$\boldsymbol{F}$: the net force exerted by the fluid on the surrounding.

Simplifying assumptions:

(1) $\boldsymbol{v}$ is taken as normal to the cross-sectional plane over the entir entrance and exit

$$\frac{d}{dt}\int_{z_1}^{z_2}\rho<\boldsymbol{v}>A\,dz = \rho_1<V^2>_1\boldsymbol{A_1}-\rho_2<V^2>_2\boldsymbol{A_2}$$
$$+p_1\boldsymbol{A_1}-p_2\boldsymbol{A_2}-\boldsymbol{F}+(\int_{z_1}^{z_2}\rho A\,dz)\boldsymbol{g}$$

(2) Let's define $\beta \equiv \dfrac{<V^2>}{<V>^2}$

For turbulent flow : $\beta \approx 1$

For laminar flow : $\beta = \dfrac{4}{3}$

$$\frac{d}{dt}\int_{z_1}^{z_2} \rho <\boldsymbol{v}> A\,dz = \beta_1 \rho_1 <V>_1^2 \boldsymbol{A_1} - \beta_2 \rho_2 <V>_2^2 \boldsymbol{A_2} + p_1 \boldsymbol{A_1} - p_2 \boldsymbol{A_2} - \boldsymbol{F} + \left(\int_{z_1}^{z_2} \rho A\,dz\right)\boldsymbol{g}$$

Since $\rho <V> A = w$, $<V>\boldsymbol{A} = <\boldsymbol{v}>A$,

$$\frac{d}{dt}\int_{z_1}^{z_2} \rho <\boldsymbol{v}> A\,dz = \beta_1 w_1 <\boldsymbol{v}>_1 - \beta_2 w_2 <\boldsymbol{v}>_2 + p_1 \boldsymbol{A_1} - p_2 \boldsymbol{A_2} - \boldsymbol{F} + \left(\int_{z_1}^{z_2} \rho A\,dz\right)\boldsymbol{g}$$

At steady state, $w_1 = w_2 = w$ and $d/dt = 0$

$$\mathbf{0} = w(\beta_1 \langle \boldsymbol{v} \rangle_1 - \beta_2 \langle \boldsymbol{v} \rangle_2) + p_1 \boldsymbol{A_1} - p_2 \boldsymbol{A_2} - \boldsymbol{F} + \left(\int_{z_1}^{z_2} \rho A \, dz \right) \boldsymbol{g}$$

Aside on spring and dashpots:

Fig. 5-7. Schematic of a mass-spring-dashpot system.

In a mass-spring-dashpot system,

spring force : $-kx$

dashpot damping force : $-\mu dx/dt$

The momentum balance is then

$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + kx = 0$$

Multiply each term by dx/dt to obtain

$$m\frac{dx}{dt}\frac{d^2x}{dt^2}+\mu\left(\frac{dx}{dt}\right)^2+kx\frac{dx}{dt}=0$$

or $$\frac{1}{2}m\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2\right]+\mu\left(\frac{dx}{dt}\right)^2+\frac{1}{2}k\frac{d(x^2)}{dt}=0$$

After the integration,

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\mu\int_0^t\left(\frac{dx}{dt}\right)^2dt+\frac{1}{2}kx^2+\text{constant}=0$$

kinetic energy	viscous damping by the dashpot	potential energy of the extended spring