Subsections

Optimization

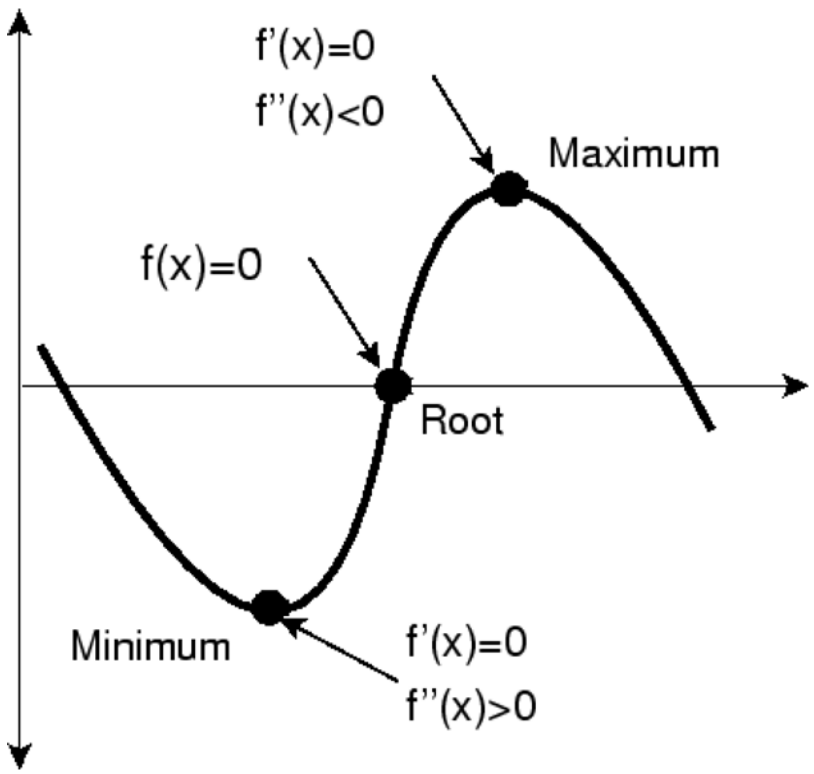

Figure 4.1: The illustation of the difference between roots and optima.

An optimization or mathematical programming problem

Find $\mathbf{x}$, which minimizes or maximizes $f(\mathbf{x})$

subject to

$$d_i(\mathbf{x}) \leq a_i \quad i = 1, 2, \ldots, m \tag{4.1}$$

$$e_i(\mathbf{x}) = b_i \quad i = 1, 2, \ldots, p \tag{4.2}$$

where $\mathbf{x}$ is an n-dimensional design vector, $f(\mathbf{x})$ is the objective function, $d_i(\mathbf{x})$ are inequality constraints, $e_i(\mathbf{x})$ are equality constraints.

Classification of optimization problem

- The form of $f(\mathbf{x})$:
  - If $f(\mathbf{x})$ and the constraints are linear, *linear programming*.
  - If $f(\mathbf{x})$ is quadratic and the constraints are linear, *quadratic programming*.
  - If $f(\mathbf{x})$ is not linear or quadratic and/or the constraints are nonlinear, *nonlinear programming*.
- For constrained problem
  - Unconstained optimization
  - Constrained optimization
- Dimensionality
  - One-dimensional problem
  - Multi-dimensional problem

One-dimensional Unconstrained Optimization

Golden-Section Search

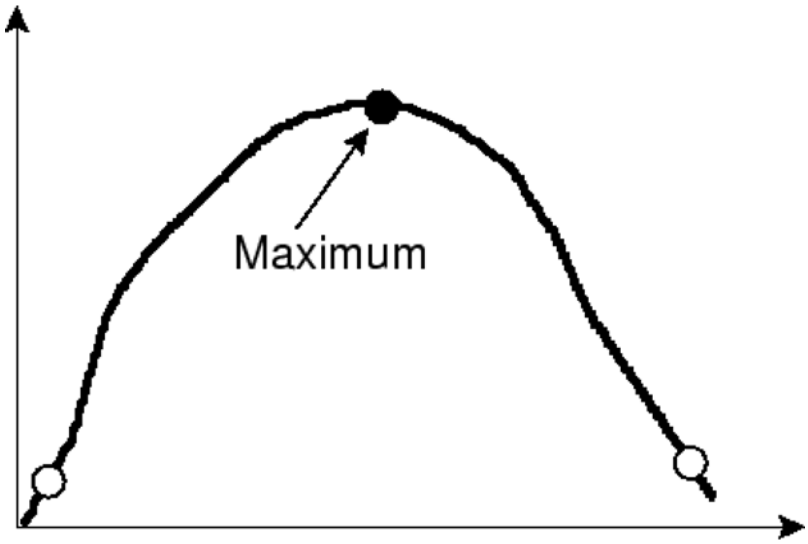

Figure 4.2: The illustation of the Golden-section search method.

Golden-section search method is similar to the bisection method in solving for the root of a single nonlinear equation. Golden-section search method can be achived by specifying that the following two conditions hold:

$$\ell_0 = \ell_1 + \ell_2 \tag{4.3}$$

$$\frac{\ell_1}{\ell_0} = \frac{\ell_2}{\ell_1} \tag{4.4}$$

Defining $R = \ell_2/\ell_1$

(4.5)

$$R = 0.61803....$$

This value is called the *golden ratio*.

Disadvantages

- Many evaluation
- Time-consuming evaluation

Quadratic Interpolation

Quadratic interpolation takes advantages of the fact that a second-order polynomial often provides a good approximation to the shape of $f(x)$ near an optimum.

An estimate of the optimal x

$$x_3 = \frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{2f(x_0)(x_1 - x_2) + 2f(x_1)(x_2 - x_0) + 2f(x_2)(x_0 - x_1)} \tag{4.6}$$

Newton's Method

At an optimum, the optimal value x^* satisfy

$$f'(x^*) = 0 \tag{4.7}$$

With a second-order Taylor series of $f(x)$, we can find the following equations for an estimate of the optimal

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} \tag{4.8}$$

Multidimensional Unconstrained Optimization

Classification of unconstrained optimization problems

- Nongradient or direct methods
- Gradient or descent methods

Direct Methods

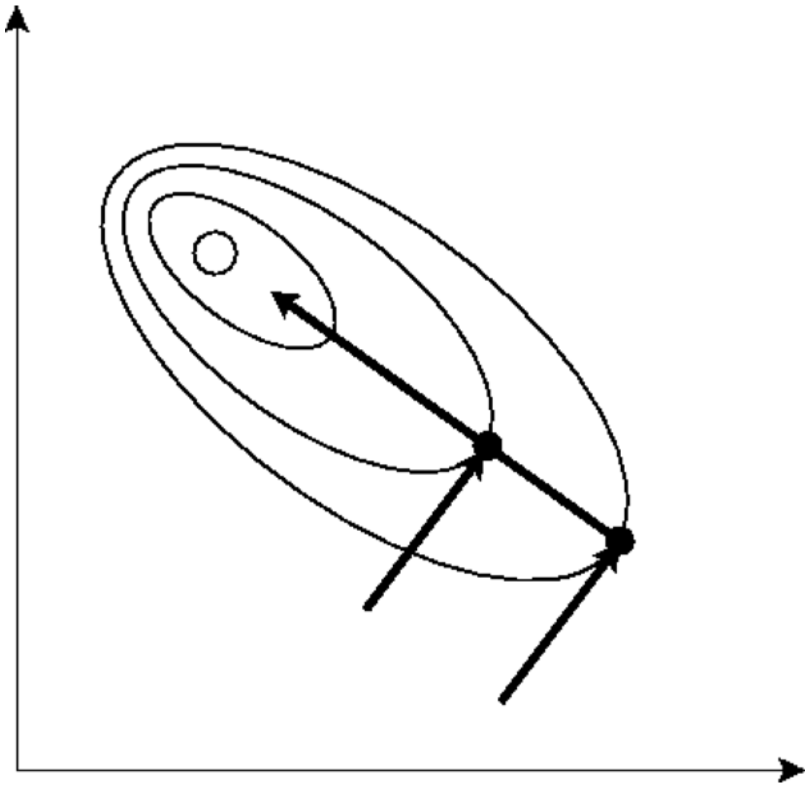

Figure 4.3: Conjugate directions.

These methods vary from simple brute force approaches to more elegant techniques that attempt to exploit the nature of the function.

- random search : repeatedly evaluates the function at randomly selected values of the independent variables.
- univariate search : change one variable at a time to improve the approximation while the other variables are held constant. Since only one variable is changed, the problem reduces to a sequence of one-dimensional searches.

Gradient Methods

Gradient methods use derivative information to generate efficient algorithms to locate optima.

The gradient is defined as

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{4.9}$$

Derivative information

- First derivative:
  - a steepest trajectory of the function
  - whether it is a optima
- Second derivative: called as Hessian, H
  - If $|H| > 0$, it is a local minimum
  - If $|H| < 0$, it is a local maximum
  - If $|H| = 0$, it is a saddle point

The quantity $|H|$ is equal to the determinant of a matrix made up of the second derivatives and, for example, the Hessian of a two-dimensional system is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

The steepest-descent algorithm is summaried as

- Determine the best direction
- Determine the best value along the search direction.

1. Calculate the partial derivatives

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_i}$$

2. Calculate the search vector

$$\mathbf{s} = -\nabla f(\mathbf{x}^k)$$

3. Use the relation

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \lambda^k \mathbf{s}^k$$

to obtain the value of $\mathbf{x}^{k+1}$. To get λ^k use the following equations

$$f(\mathbf{x}^{k+1}) = f(\mathbf{x}^k + \lambda \mathbf{x}^k) = f(\mathbf{x}^k) + \nabla^T f(\mathbf{x}^k) \lambda \mathbf{s}^k + \frac{1}{2}(\lambda \mathbf{s}^k)^T \mathbf{H}(\mathbf{x}^k)(\lambda \mathbf{s}^k)$$

To get the minimum, differentiate with respect to λ and equate the derivative to zero

$$\frac{df(\mathbf{x}^k + \lambda \mathbf{x}^k)}{d\lambda} = \nabla^T f(\mathbf{x}^k)\mathbf{s}^k + (\mathbf{s}^k)^T \mathbf{H}(\mathbf{x}^k)(\lambda \mathbf{s}^k)$$

with the result

$$\lambda^{\text{opt}} = \frac{\nabla^T f(\mathbf{x}^k)\mathbf{s}^k}{(\mathbf{s}^k)^T \mathbf{H}(\mathbf{x}^k)\mathbf{s}^k}$$

Contrained Optimization

Linear Programming

Four general outcome from linear programming

- Unique solution
- Alternate solutions

- No feasible solution
- Unbounded problems

Optimization with Packages

- Matlab:
  - fmin : Minimize function of one variable
  - fmins : Minimiza function of several varaibles
  - fsolve : Solve nonlinear equations by a least squares method
- IMSL : various routines are exist to solve optimization problems

Engineering Applications: Optimization

See the textbook

Taechul Lee
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