Subsections

Curve Fitting

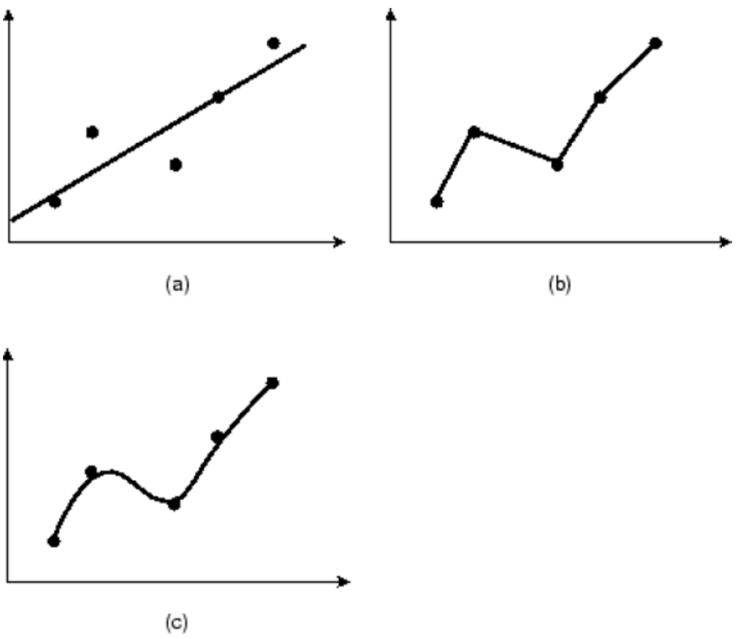

Figure 5.1: Three attempts to fit a best curve.

The simplest method for fitting a curve to data is to plot the points and then sketch a line

- (a) Characterize the general upward trend of the data with a straight line
- (b) Use straight-line segment or linear interpolation
- (c) Use curves to try to captuer the meanderings

Simple statistics

- Arithmetic mean

$$\bar{y} = \frac{\sum y_i}{n}$$

- Standard deviation : the measure of spread of a sample

$$s_y = \sqrt{\frac{S_t}{n-1}}$$

where S_t is the total sum of the squares of the residual between the data points and the mean, or

$$S_t = \sum (y_i - \bar{y})^2$$

- Variance : The square of the standard deviation

$$s_y^2 = \frac{S_t}{n-1}$$

- Coefficient of variation (c.v.) : The spread of data

$$c.v. = \frac{s_y}{\bar{y}} 100\%$$

Least-Squares Regression

Lest-squares regression is drived from a curve that minimized the discrepancy between the data points and the curve.

Linear Regression

A least-squares approximation is fitting a straight line to a set of paired observation. The mathematical expression for the straight line is

$$y = a_0 + a_1 x + e \tag{5.1}$$

The error, or residual, is the discrepancy between the true value of y and the approximate value, $a_0 + a_1 x$ and that is

$$e = y - a_0 + a_1 x \tag{5.2}$$

The criterion for least-squares regression is

$$\min S_r = \sum_i^n e_i^2 = \sum_i^n (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_i^n (y_i - a_0 - a_1 x_i)^2 \tag{5.3}$$

To determine values of a_0 and a_1, differentiate (5.3)

$$\frac{\partial S_r}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i) \tag{5.4}$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum \left[(y_i - a_0 - a_1 x_i) x_i\right] \tag{5.5}$$

And setting these derivatives equal to zero, we get the so-called normal equations

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i \tag{5.6}$$

$$0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2 \tag{5.7}$$

The coefficients of a straight line are

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \tag{5.8}$$

$$a_0 = \bar{y} - a_1 \bar{x} \tag{5.9}$$

Quantification of error of linear regression

- The sum of the square of the residual
    - A sampled data system

$$S_t = \sum (y_i - \bar{y})^2$$

    - A linear regressioned system

$$S_r = \sum (y_i - a_0 - a_1 x_i)^2$$

- Standard deviation
    - A sampled data system

$$s_y = \sqrt{\frac{S_t}{n-1}}$$

s_y quantifies the spread around mean.

    - A linear regressioned system

$$s_{y/x} = \sqrt{\frac{S_r}{n-2}}$$

$s_{y/x}$ quantifies the spread around the regression line.

- The goodness of a fit

$$r^2 = \frac{S_t - S_r}{S_t}$$

where r^2 is called the coefficient of determination and r is the correlation coefficient.

See the figure 17.4 in the textbook

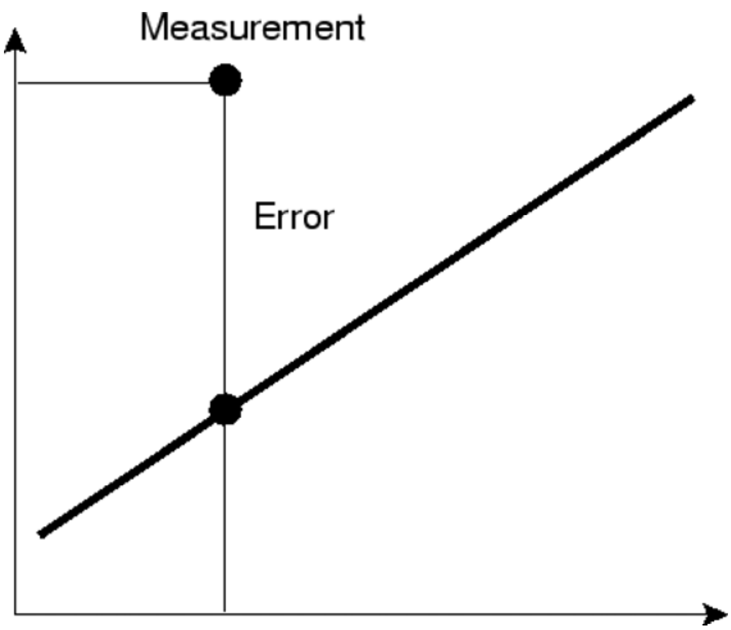

Figure 5.2: The residual in linear regression

General Linear Least-Squares

The general linear least-square model:

$$y = a_0 z_0 + a_1 z_1 + \cdots + a_m z_m + e \tag{5.15}$$

In matrix notation

$$Y = ZA + E \tag{5.16}$$

Note that Z is not a square matrix but we want to know about A.

$$Z^T Z A = Z^T Y \tag{5.17}$$

Now A is

$$A = (Z^T Z)^{-1} Z^T Y \tag{5.18}$$

Nonlinear Regression

Gauss-Newton method

1. Use a Taylor series to linearize a nonlinear function
2. Apply least-square theorie to obtain new estimate of the parameters that move in the direction of minimizing the residual.

Interpolation

Newton's Divided-Difference Interpolating Polynomials

Linear interpolation : connect two data points with a straight line

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \tag{5.19}$$

Quadratic interpolation : connect three data points with a second-order polynomial

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{5.20}$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Newton's interpolating polynomial : connect $n+1$ data with nth-order polynomial

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)\cdots(x - x_{n-1}) \tag{5.21}$$

where the coefficients are

$$\begin{aligned} b_0 &= f(x_0) \\ b_1 &= f[x_1, x_0] \\ &\vdots \\ b_n &= f[x_n, x_{n-1}, \ldots, x_0] \end{aligned}$$

where the bracket function evaluations are finite divided differences.

nth finite divided difference is

$$f[x_n, x_{n-1}, \ldots, x_0] = \frac{f[x_n, \ldots, x_1] - f[x_{n-1}, \ldots, x_0]}{x_n - x_0} \tag{5.22}$$

Newton's divided-difference interpolating polynomial is

$$f_n(x) = f(x_0) + (x - x_0)f[x_1, x_0] + \cdots (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_n, \ldots, x_0] \tag{5.23}$$

Lagrange Interpolating Polynomial

The Lagrange interpolating polynomial is simply a reformulation of the Newton polynomial that avoids the computation of divided differences.

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i) \tag{5.24}$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \tag{5.25}$$

where $\prod$ designates the ``product of."

Spline Interpolation

Spline interpolation is an alternative approach that lower-order polynomial is applied to subsets of data point. Especially, when third-order curves are employed to connect each pair of data points, it is called cubic spline.

Linear splines : the simplest connection between two points is a straight line.

$$\begin{aligned}
f(x) &= f(x_0) + m_0(x - x_0) && x_0 \leq x \leq x_1 \\
f(x) &= f(x_1) + m_1(x - x_1) && x_1 \leq x \leq x_2 \\
&\vdots \\
f(x) &= f(x_{n-1}) + m_1(x - x_{n-1}) && x_{n-1} \leq x \leq x_n
\end{aligned}$$

where m_i is the slope of the straight line

$$m_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \tag{5.26}$$

Quadratic splines : connect three points with second-order polynomials.

- The function values of adjacent polynomials must be equal at the interior knots.
- The first and last functions must pass through the end points.
- The first derivatives at the interior knots must be equal.
- Assume that the second derivative is zero at the first point.

Cubic splines : derive a third-order polynomial for each interval between knots

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \tag{5.27}$$

Fourier Approximation

In early 1800s, the French mathematician Fourier proposed that ``any function can be represented by an infinite sum of sine and cosine terms." There are functions that do not have a representation as a Fourier series, however, most functions can be so represented. Fourier approximation is another representation of a function with trigonometric series.

Trigonometric identities

- $\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$
- $\sin A \cos B = \frac{1}{2}\left[\sin(A - B) + \sin(A + B)\right]$
- $\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

Fourier series

Assume that $f(x)$ is a periodic function of period 2π and is integrable over a period.

$$f(x) \simeq A_0 + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)] \tag{5.28}$$

- A_0 : integrating on both sides of (5.28) from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} A_0 dx + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos(nx)dx + \sum_{n=1}^{\infty} B_n \int_{-\pi}^{\pi} \sin(nx)dx$$

The last two integrations of trigonometric terms are equal to zero. Hence

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

- A_n : multiply both sides of (5.28) by $\cos(mx)$ and integrate

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx) f(x)dx &= \int_{-\pi}^{\pi} A_0 \cos(mx)dx \\ &+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n \cos(nx)\cos(mx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} B_n \sin(nx)\cos(mx)dx \end{aligned} \tag{5.1}$$

The only nonzero term on the right is when $m = n$ in the first summation

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx$$

- B_n : multiply both sides of (5.28) by $\sin(mx)$ and integrate

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) f(x)dx &= \int_{-\pi}^{\pi} A_0 \sin(mx)dx \\ &+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n \cos(nx)\sin(mx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} B_n \sin(nx)\sin(mx)dx \end{aligned} \tag{5.2}$$

The only nonzero term on the right is when $m = n$ in the second summation

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Fourier series for any period $p = 2L$

Consider the function whose period is $p = 2L$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi}{L} x + B_n \sin \frac{n\pi}{L} x \right) \tag{5.33}$$

where the Fourier coefficients of $f(x)$ are given by the Euler formulas

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\ A_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx \\ B_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx \end{aligned} \tag{5.34}$$

Fourier series for even and odd functions

- Even function:

$$g(-x) = g(x)$$

  And integral value of a even function is

$$\int_{-L}^{L} g(x) dx = 2 \int_{0}^{L} g(x) dx$$

- Odd function:

$$h(-x) = -h(x)$$

  And integral value of a even function is

$$\int_{-L}^{L} h(x) dx = 0$$

- Fourier cosine series: the Fourier series of an even function of period $2L$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x$$

- Fourier sine series: the Fourier series of an odd function of period $2L$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

Complex form of Fourier series : Real sines and cosines can be expressed in terms of complex exponentials by the formulas

(5.41)

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$
$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

From this

$$A_n \cos nx + B_n \sin nx = \frac{1}{2}A_n(e^{inx} + e^{-inx}) + \frac{1}{2i}B_n(e^{inx} - e^{-inx}) \tag{5.42}$$

$$= \frac{1}{2}(A_n - iB_n)e^{inx} + \frac{1}{2}(A_n + iB_n)e^{-inx} \tag{5.43}$$

$$= c_n e^{inx} + c_{-n} e^{-inx} \tag{5.44}$$

With the above equation

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} \tag{5.45}$$

where

$$c_n = A_n - iB_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos(nx) - i\sin(nx))dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

This is the so-called complex form of the Fourier series, or complex Fourier series of $f(x)$.

Sinusoidal function : represent any waveform with a sine or cosine

$$f(t) = A_0 + C_1 \cos(\omega_0 t + \theta) \tag{5.46}$$

where A_0 is the mean value, C_1 is the amplitude, ω_0 is the angular frequency, and θ is the phase angle or phase shift.

The angular frequency is related to frequency f (in cycles/time)

$$\omega_0 = 2\pi f \tag{5.47}$$

and frequency is

$$f = \frac{1}{T} \tag{5.48}$$

The trigonometric identity gives

$$f(t) = A_0 + A_1 \cos(\omega_0 t) + B_1 \sin(\omega_0 t) \tag{5.49}$$

where $A_1 = C_1 \cos(\theta)$, $B_1 = -C_1 \sin(\theta)$

Curve Fitting with Sinusoidal Functions

Least-squares fit of a sinusoidal function is to determine coefficient values that minimize

$$S_r = \sum_{i=1}^{N} \{y_i - [A_0 + A_1 \cos(\omega_0 t_i) + B_1 \sin(\omega_0 t_i)]\}^2 \tag{5.50}$$

$$\begin{bmatrix} N & \sum \cos(\omega_0 t) & \sum \sin(\omega_0 t) \\ \sum \cos(\omega_0 t) & \sum \cos^2(\omega_0 t) & \sum \cos(\omega_0 t) \sin(\omega_0 t) \\ \sum \sin(\omega_0 t) & \sum \cos(\omega_0 t) \sin(\omega_0 t) & \sum \sin^2(\omega_0 t) \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} \sum y \\ \sum y \cos(\omega_0 t) \\ \sum y \sin(\omega_0 t) \end{Bmatrix} \tag{5.51}$$

For equispaced system

$$\int_0^T \cos(\omega_0 t) dt = -\frac{1}{\omega_0} \sin(\omega_0 t) \Big|_0^T = 0 \tag{5.52}$$

where $\omega_0 T = \frac{2\pi}{T} T = 2\pi$. These relationhips give

$$\begin{bmatrix} N & 0 & 0 \\ 0 & N/2 & 0 \\ 0 & 0 & N/2 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} \sum y \\ \sum y \cos(\omega_0 t) \\ \sum y \sin(\omega_0 t) \end{Bmatrix} \tag{5.53}$$

or

$$A_0 = \frac{\sum y}{N} \tag{5.54}$$

$$A_1 = \frac{2}{N} \sum y \cos(\omega_0 t) \tag{5.55}$$

$$A_2 = \frac{2}{N} \sum y \sin(\omega_0 t) \tag{5.56}$$

The above equations are similar with the determination of Fourier series.

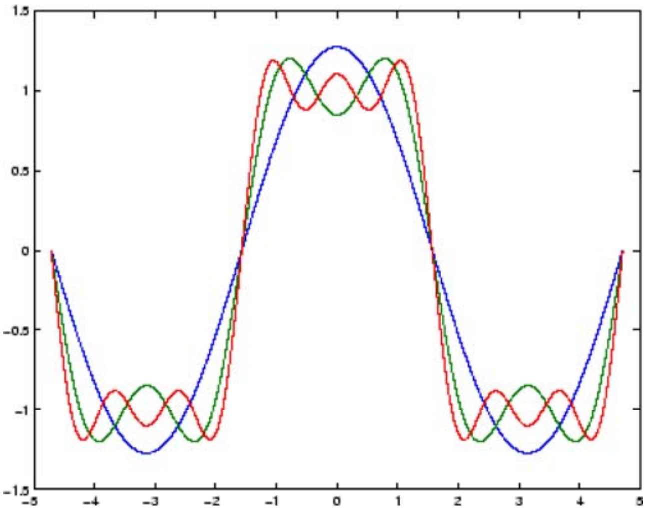

Figure 5.3: The Fourier series approximation of the square wave.

Fourier Integral and Transform

Some of phenomenon does not occured repeatedly or it will be a long time until it occurs again. In this case we use Fourier integral that can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The transition from a periodic to a nonperiodic function can be effected by allowing the period to approach infinity. In other words, as T becomes infinite, the function never repeats itself and thus becomes aperiodic.

From Fourier series to the Fourier intergral

Consider any periodic function $f_L(x)$ of period $2L$

$$f_L(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos \omega_n x + B_n \sin \omega_n x) \tag{5.57}$$

where $\omega_n = n\pi/L$. Insert A_n and B_n which are given by the Euler formulas.

$$\begin{aligned} f_L(x) = &\frac{1}{2L}\int_{-L}^{L} f_L(v)dv \\ &+ \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos \omega_n x \int_{-L}^{L} f_L(v)\cos \omega_n v dv + \sin \omega_n x \int_{-L}^{L} f_L(v)\sin \omega_n v dv\right] \end{aligned} \tag{5.3}$$

Now set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \tag{5.58}$$

Then $1/L = \Delta\omega/\pi$, and

$$\begin{aligned} f_L(x) = &\frac{1}{2L}\int_{-L}^{L} f_L(v)dv \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos \omega_n x\Delta\omega \int_{-L}^{L} f_L(v)\cos \omega_n v dv + \sin \omega_n x\Delta\omega \int_{-L}^{L} f_L(v)\sin \omega_n v dv\right] \end{aligned} \tag{5.4}$$

Let $L \longrightarrow \infty$ and assume a periodic function $f_L(x)$ to be a aperiodic function.

$$f(x) = \lim_{L \to \infty} f_L(x) \tag{5.59}$$

Then $1/L \longrightarrow 0$ and the first term of function approaches zero.

$$f_L(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos \omega_n x \Delta\omega \int_{-L}^{L} f_L(v) \cos \omega_n v dv + \sin \omega_n x \Delta\omega \int_{-L}^{L} f_L(v) \sin \omega_n v dv \right] \tag{5.60}$$

$L \longrightarrow 0$ results in $\Delta\omega \to 0$ and the sum of infinite series become an integral from 0 to ∞.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos \omega x \int_{-\infty}^{\infty} f(v) \cos \omega v dv + \sin \omega x \int_{-\infty}^{\infty} f(v) \sin \omega v dv \right] d\omega \tag{5.61}$$

Introduce $A(\omega)$ and $B(\omega)$ as

$$A(\omega) = \int_{-\infty}^{\infty} f(v) \cos \omega v dv, \quad B(\omega) = \int_{-\infty}^{\infty} f(v) \sin \omega v dv \tag{5.62}$$

Finally Fourier series for an aperiodic equation become

$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] \, d\omega \tag{5.63}$$

This is called a representation of $f(x)$ by a Fourier integral.

Alternatively, the Fourier integral can be written as complex Fourier series.

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} c_n e^{i\omega_n x} \\ c_n &= \frac{1}{2L} \int_{-L}^{L} f(u) e^{-i\omega_n u} du \end{aligned} \tag{5.64}$$

$$f(x) = \sum_{-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} f(u) e^{-i\omega_n u} du \right] e^{i\omega_n x} \tag{5.65}$$

Use $1/L = \Delta\omega/\pi$

$$f(x) = \sum_{-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(u) e^{-i\omega_n u} du \right] e^{inx} \tag{5.66}$$

(5.67)

$$= \sum_{-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(u) e^{i\omega_n(x-u)} du = \frac{1}{2\pi} \sum_{-\infty}^{\infty} F(\omega_n) \Delta\omega$$

where

$$F(\omega_n) = \int_{-L}^{L} f(u) e^{i\omega_n(x-u)} du \tag{5.68}$$

If $\Delta\omega$ goes to zero, a limit of a sum becomes an integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\omega(x-u)} du d\omega \tag{5.69}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} d\omega \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \tag{5.70}$$

Define $g(\omega)$ by

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \tag{5.71}$$

Then

$$f(x) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} d\omega \tag{5.72}$$

Fourier Transform

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} d\omega \\ g(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \end{aligned} \tag{5.73}$$

$f(x)$ and $g(\omega)$ are called a pair of Fourier transforms. Usually, $g(\omega)$ is called the Fourier transform of $f(x)$, and $f(x)$ is called the inverse Fourier transform of $g(\omega)$.

Discrete Fourier Transform (DFT)

In engineering, functions are often represented by finite sets of discrete values and data is often collected in or converted to such a discrete format. For the discrete time system, a discrete Fourier transform can be written as

$$F_k = \sum_{n=0}^{N-1} f_n e^{-i\omega_0 n} \tag{5.74}$$

and the inverse Fourier transform as

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{i\omega_0 n} \tag{5.75}$$

where $\omega_0 = 2\pi/N$.

Fast Fourier Transform (FFT)

The fast Fourier transform (FFT) is an algorithm that has been developed to compute the DFT in an extremely economical fashion.

The Power Spectrum

A power spectrum is developed from the Fourier transform and it is derived from the analysis of the power output of electrical systems. The power of a periodic signal can be defined as

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt \tag{5.76}$$

A power spectrum can be calculated by the power associated with each frequency component.

Curve Fitting with Libraries and Packagies

- Matlab:
  - polyfit
  - polyval
  - poly2sym
  - interp1
  - spline
  - fft
- IMSL: various routines are exist to solve curve fitting and fft problems

Engineering Applications: Curve Fitting

See the textbook

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