CHE302 LECTURE VI
DYNAMIC BEHAVIORS OF REPRESENTATIVE PROCESSES

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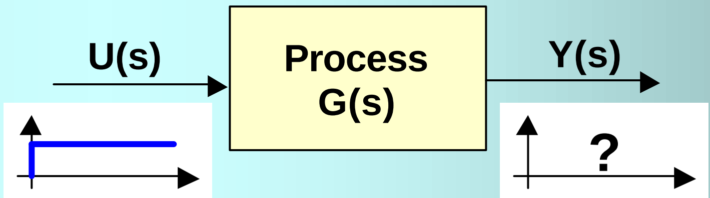
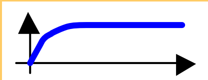
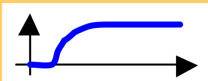
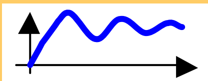
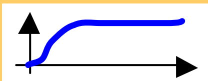
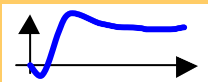
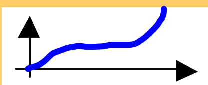
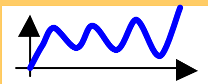
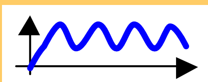
REPRESENTATIVE TYPES OF RESPONSE

- **For step inputs**



Y(t)	Type of Model, G(s)
	Nonzero initial slope, no overshoot or nor oscillation, 1^{st} order model
	1^{st} order+Time delay
	Underdamped oscillation, 2^{nd} or higher order
	Overdamped oscillation, 2^{nd} or higher order
	Inverse response, negative (RHP) zeros
	Unstable, no oscillation, real RHP poles
	Unstable, oscillation, complex RHP poles
	Sustained oscillation, pure imaginary poles

1^{ST} ORDER SYSTEM

- **First-order linear ODE (assume all deviation variables)**

$$t\frac{dy(t)}{dt} = -y(t) + Ku(t) \xrightarrow{L} (t s + 1)Y(s) = KU(s)$$

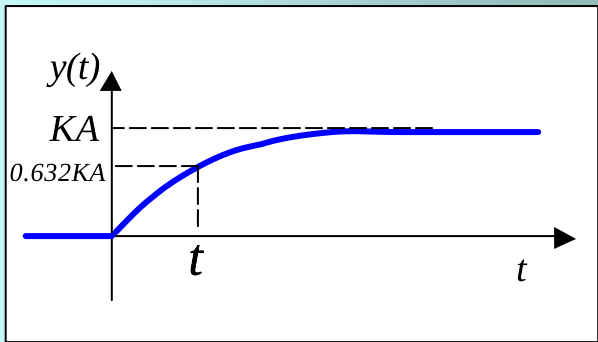
- **Transfer function:**

$$\frac{Y(s)}{U(s)} = \frac{K}{(t s + 1)}$$

Gain

Time constant

- **Step response:**



With $U(s) = A/s$,

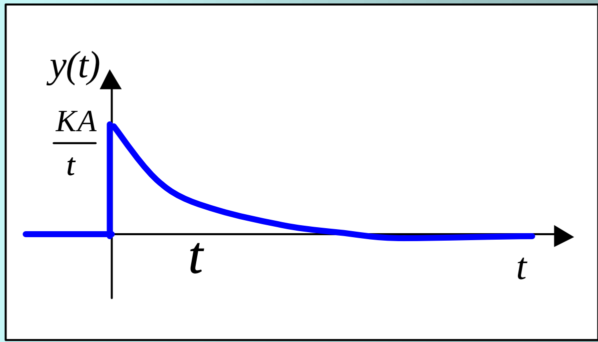
$$Y(s) = \frac{KA}{s(t s + 1)} \xrightarrow{L} y(t) = KA(1 - e^{-t/t})$$

- $y(t) = KA(1 - e^{-t/t}) \approx 0.632KA$
- $KA(1 - e^{-t/t}) \geq 0.99KA \Rightarrow t \approx 4.6t$ (Settling time=$4t \sim 5t$)
- $y'(0) = KAe^{-t/t}/t\big|_{t=0} = KA/t \neq 0$ (Nonzero initial slope)

- **Impulse response**

With $U(s) = A,$

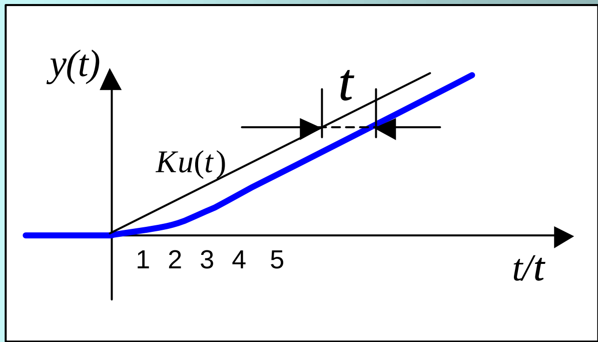
$$Y(s) = \frac{KA}{(t s + 1)} \xrightarrow{\mathrm{L}} y(t) = \frac{KA}{t} e^{-t/t}$$



- **Ramp response**

With $U(s) = a / s^2,$

$$Y(s) = \frac{Ka}{s^2 (t s + 1)} \xrightarrow{\mathrm{L}} y(t) = Kat\, e^{-t/t} + Ka(t - t)$$

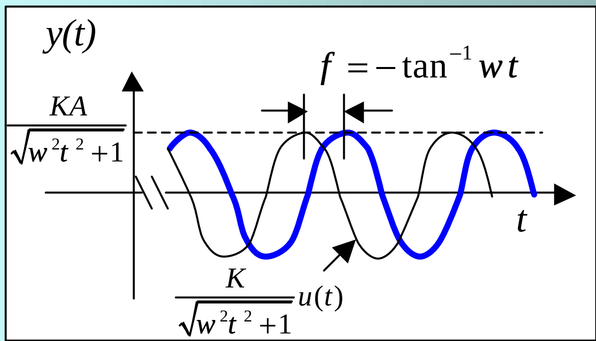


- **Sinusoidal response**

With $U(s) = \mathrm{L}\left[A \sin wt \right] = w / (s^2 + w^2),$

$$Y(s) = \frac{KAw}{(t s + 1)(s^2 + w^2)} \xrightarrow{\mathrm{L}}$$

$$y(t) = \frac{KA}{w^2 t^2 + 1} (wt\, e^{-t/t} - wt \cos wt + \sin wt)$$



- **Ultimate sinusoidal response** $(t \to \infty)$

$$y_\infty(t) = \lim_{t\to\infty} \frac{KA}{w^2 t^2 + 1}(wt e^{-t/t} - wt \cos wt + \sin wt)$$

(an arrow marks $e^{-t/t}$ → 0)

$$= \frac{KA}{w^2 t^2 + 1}(-wt \cos wt + \sin wt)$$

$$= \frac{KA}{\sqrt{w^2 t^2 + 1}} \sin(wt + f) \qquad (f = -\tan^{-1} wt)$$

Amplitude **Phase angle**

  - **The output has the same period of oscillation as the input.**
  - **But the amplitude is attenuated and the phase is shifted.**

Normalized Amplitude Ratio (AR_N) $= \frac{1}{\sqrt{w^2 t^2 + 1}} < 1$ **Phase angle** $= -\tan^{-1} wt$

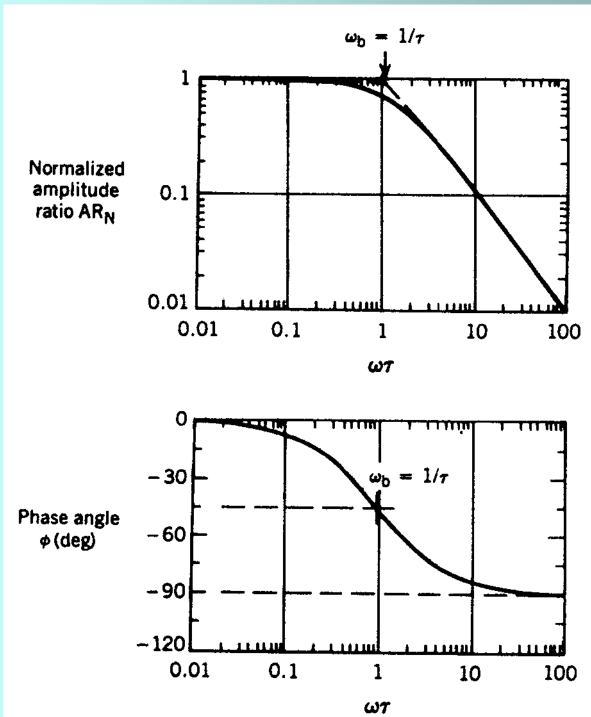
  - **High frequency input will be attenuated more and phase is shifted more.**

BODE PLOT FOR 1ST ORDER SYSTEM

- **AR plot asymptote**

$$AR_N(w \to 0) = \lim_{w\to 0} \frac{1}{\sqrt{w^2 t^2 + 1}} = 1$$

$$AR_N(w \to \infty) = \lim_{w\to \infty} \frac{1}{\sqrt{w^2 t^2 + 1}} = \frac{1}{wt}$$



- **Phase plot asymptote**

$$f(w \to 0) = -\lim_{w\to 0} \tan^{-1} wt = 0^o$$

$$f(w \to \infty) = -\lim_{w\to \infty} \tan^{-1} wt = -90^o$$

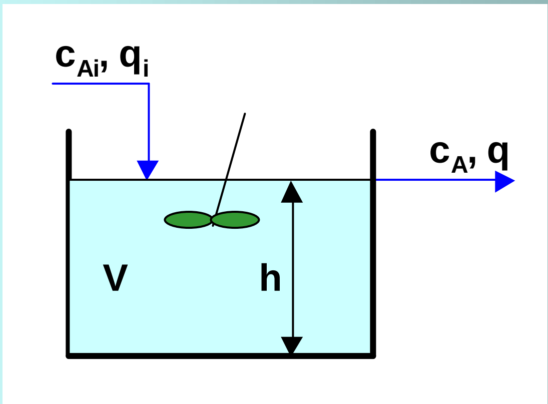
- **It is also called "low-pass filter"**

1^{ST} ORDER PROCESSES

- **Continuous Stirred Tank**

$$V\frac{dc_A}{dt} = qc_{Ai} - qc_A$$

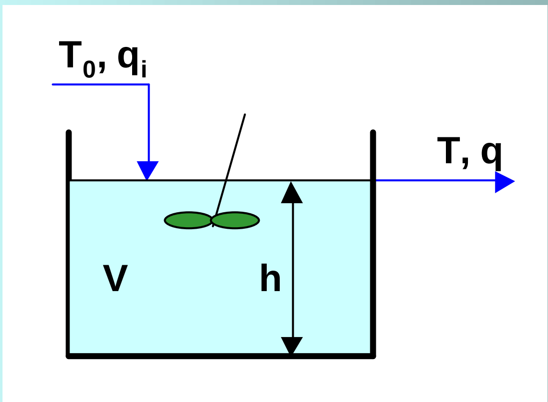
$$\frac{C_A(s)}{C_{Ai}(s)} = \frac{q}{Vs+q} = \frac{1}{(V/q)s+1}$$



- **With constant heat capacity and density**

$$rVC_p\frac{d(T-T_{ref})}{dt} = r\,qC_p(T_0-T_{ref}) - r\,qC_p(T-T_{ref})$$

$$\frac{T(s)}{T_0(s)} = \frac{q}{Vs+q} = \frac{1}{(V/q)s+1}$$



INTEGRATING SYSTEM

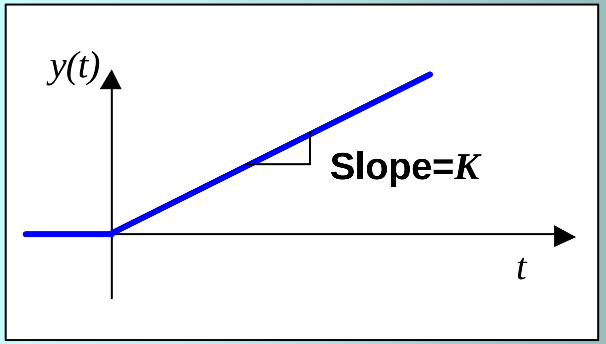
- $$\frac{dy(t)}{dt} = Ku(t) \xrightarrow{\mathcal{L}} sY(s) = KU(s)$$

- **Transfer Function:** $$\frac{Y(s)}{U(s)} = \frac{K}{s}$$

- **Step Response**

  With $U(s) = 1/s$,

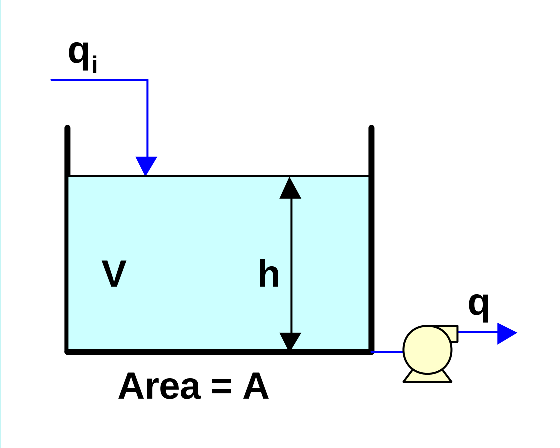
  $$Y(s) = \frac{K}{s^2} \xrightarrow{\mathcal{L}} y(t) = Kt$$



  - **The output is an integration of input.**
  - **Impulse response is a step function.**
  - **Non self-regulating system**

INTEGRATING PROCESSES

- **Storage tank with constant outlet flow**
  - **Outlet flow is pumped out by a constant-speed, constant-volume pump**
  - **Outlet flow is not a function of head.**



$$A\frac{dh}{dt} = q_i - q$$

$$\frac{H(s)}{Q_i(s)} = \frac{1}{As} \qquad \frac{H(s)}{Q(s)} = -\frac{1}{As}$$

2^{ND} ORDER SYSTEM

- **2^{nd} order linear ODE**

$$t^2 \frac{d^2 y(t)}{dt^2} + 2zt \frac{dy(t)}{dt} + y(t) = Ku(t) \xrightarrow{L} (t^2 s^2 + 2zt s + 1)Y(s) = KU(s)$$

- **Transfer Function:**

$$\frac{Y(s)}{U(s)} = \frac{K}{(t^2 s^2 + 2zt s + 1)}$$

Gain (→ K)

Time constant (→ t)

Damping Coefficient (→ z)

- **Step response**
  - **Varies with the type of roots of denominator of the TF.**
    - **Real part of roots should be negative for stability:** $z \geq 0$
    - **Two distinct real roots (** $z > 1$ **): overdamped (no oscillation)**
    - **Double root (** $z = 1$ **): critically damped (no oscillation)**
    - **Complex roots (** $0 \leq z < 1$ **): underdamped (oscillation)**

- **Case I** $(\zeta > 1)$ **with** $U(s)=1/s$

$$Y(s)=\frac{K}{s(\tau^2 s^2+2\zeta\tau s+1)}=\frac{K}{s(\tau_1 s+1)(\tau_2 s+1)} \xrightarrow{L} y(t)=K\left(1-\frac{\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}}{(\tau_1-\tau_2)}\right)$$

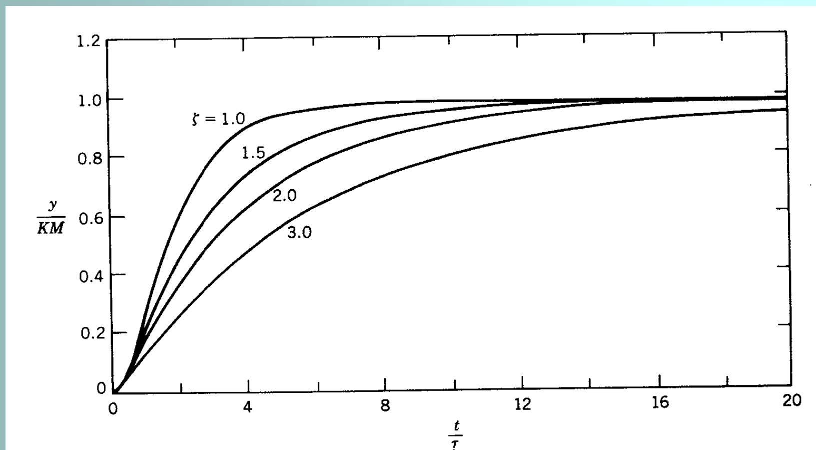
- **Case II** $(\zeta = 1)$

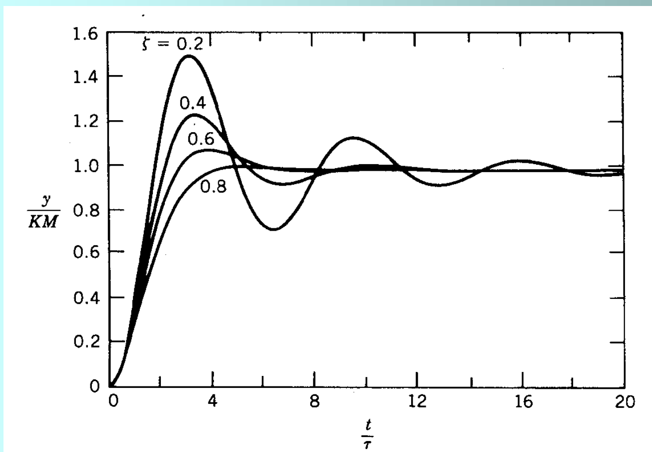
$$Y(s)=\frac{K}{s(\tau^2 s^2+2\tau s+1)}=\frac{K}{s(\tau s+1)^2} \xrightarrow{L} y(t)=K\left[1-\left(1+t/\tau\right)e^{-t/\tau}\right]$$

- **Case III** $(0 \le \zeta < 1)$

Natural frequency

$$Y(s)=\frac{K}{s(\tau^2 s^2+2\tau s+1)} \xrightarrow{L} y(t)=K\left[1-e^{-\zeta t/\tau}\left\{\cos\alpha t+\frac{\zeta}{\alpha\tau}\sin\alpha t\right\}\right] \quad \left(\alpha=\frac{\sqrt{1-\zeta^2}}{\tau}\right)$$





- **Ultimate sinusoidal response**

With $U(s) = \mathrm{L}\left[A \sin wt\right]$,

$$Y(s) = \frac{KAw}{(t^2 s^2 + t s + 1)(s^2 + w^2)} \xrightarrow{\mathrm{L}}$$

$$y(t) = \frac{KA}{\sqrt{(1 - w^2 t^2)^2 + (2zwt)^2}} \sin(wt + f) \qquad (f = -\tan^{-1}\frac{2zwt}{1 - w^2 t^2})$$

  - **Other method to find ultimate sinusoidal response**

For $(s + a + jw)$, $y(t)$ has $e^{-(a + jw)t}$ and it becomes e^{-jwt} as $t \to \infty$ $(a > 0)$.

$$G(s) = \frac{K}{(t^2 s^2 + 2zt s + 1)} \xrightarrow{s \to jw} G(jw) = \frac{K}{(1 - t^2 w^2) + 2jztw}$$

$$AR = \left|G(jw)\right| = \left|\frac{K}{(1 - t^2 w^2) + jtw}\right| = \frac{K}{\sqrt{(1 - w^2 t^2)^2 + (2zwt)^2}}$$

$$f = \measuredangle G(jw) = \tan^{-1}\frac{\mathrm{Im}(G(jw))}{\mathrm{Re}(G(jw))} = -\tan^{-1}\frac{2zwt}{1 - w^2 t^2}$$

BODE PLOT FOR 2^ND ORDER SYSTEM

- **AR plot**

$$AR_N(w \to \infty) = \lim_{w\to\infty} \frac{1}{\sqrt{(1-w^2t^2)^2 + (2zwt)^2}} = \frac{1}{(wt)^2}$$

- **Phase plot**

$$f(w \to \infty) = -\lim_{w\to\infty} \tan^{-1}\frac{2zwt}{1-w^2t^2} = \lim_{w\to\infty} \tan^{-1}\frac{-2z}{-wt} = -180^{o}$$

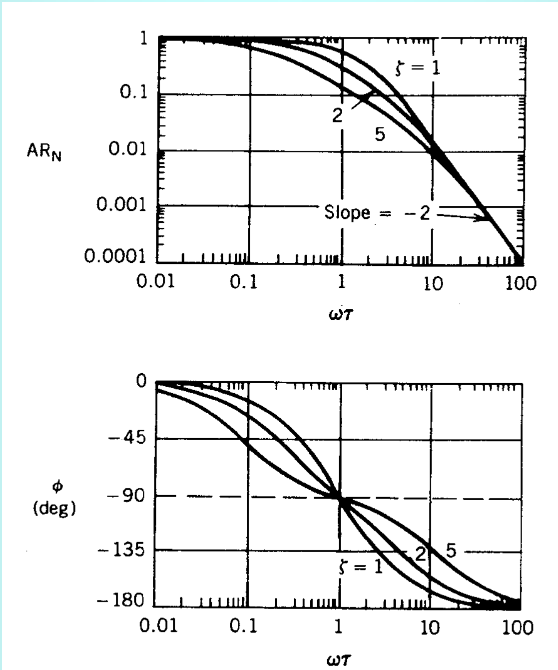
- **Resonance**

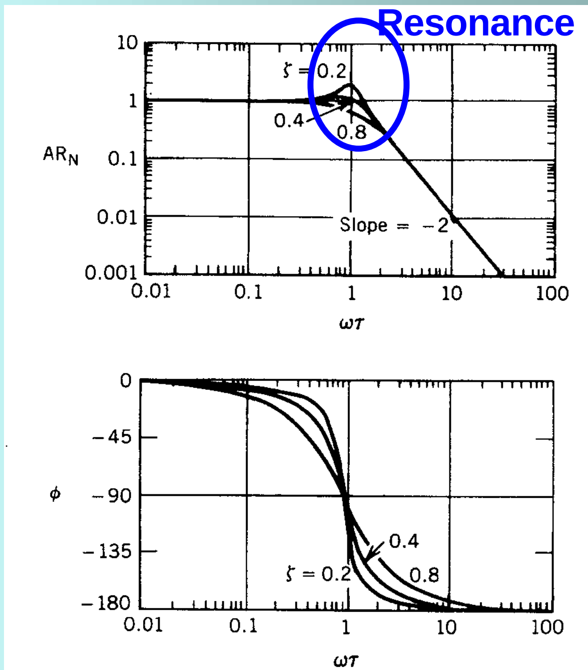
$$d(AR_N)/dw = 0$$

$$w_{\max} = \frac{\sqrt{1-2z^2}}{t}$$

$$\text{for } 0 < z < 0.707$$

The amplitude of output oscillation is bigger than that of input when the resonance occurs .





1ST ORDER VS. 2ND ORDER (OVERDAMPED)

- **Initial slope of step response**

$$\text{1st order: } y'(0) = \lim_{s\to\infty}\left\{s^2 Y(s)\right\} = \lim_{s\to\infty}\frac{KAs}{t\,s+1} = \frac{KA}{t} \neq 0$$

$$\text{2nd order: } y'(0) = \lim_{s\to\infty}\left\{s^2 Y(s)\right\} = \lim_{s\to\infty}\frac{KAs}{t^2 s + 2z t\,s + 1} = 0$$

- **Shape of the curve (Convexity)**

$$\text{1st order: } y''(t) = -Ke^{-t/t} < 0 \ \ (\text{For } K > 0) \ \Rightarrow \ \text{No inflection}$$

$$\text{2nd order: } y''(t) = -\frac{KA}{t_1 - t_2}\left(\frac{e^{-t/t_1}}{t_1} - \frac{e^{-t/t_2}}{t_2}\right)$$

$$(+ \rightarrow - \text{ as t} \uparrow) \ \Rightarrow \ \text{Inflection}$$

CHARACTERIZATION OF SECOND ORDER SYSTEM

- **2nd order Underdamped response**
  - **Rise time (t_r)**

    $$t_r = t\,(np - \cos^{-1} z\,)\,/\sqrt{1-z^{\,2}} \quad (n=1)$$

  - **Time to 1st peak (t_p)**

    $$t_p = tp\;/\sqrt{1-z^{\,2}}$$

  - **Settling time (t_s)**

    $$t_s \approx -t\;/z\;\ln(0.05)$$

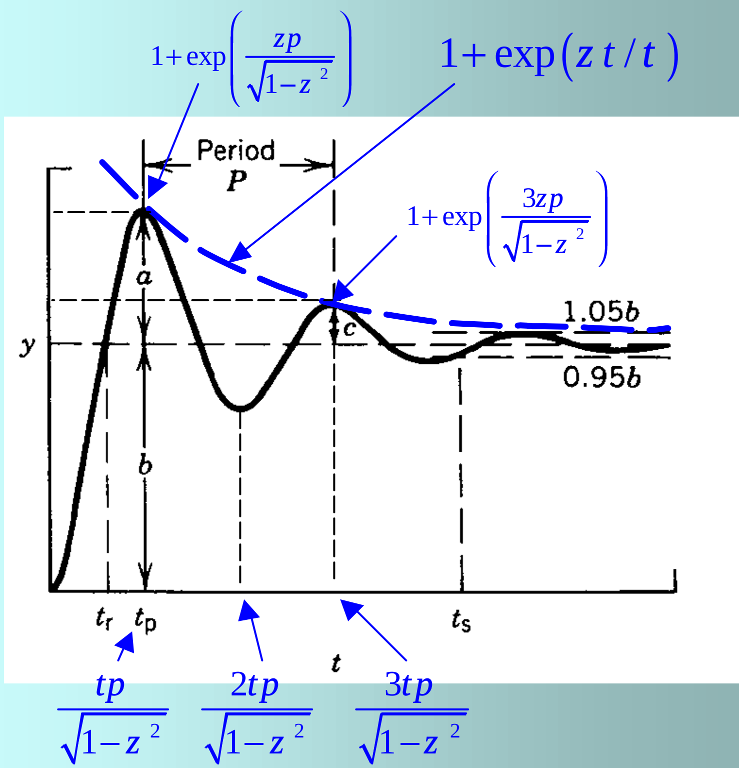
  - **Overshoot (OS)**

    $$OS = a/b = \exp\left(-pz\;/\sqrt{1-z^{\,2}}\right)$$

  - **Decay ratio (DR): a function of damping coefficient only!**

    $$DR = c/a = (OS)^2 = \exp\left(-2pz\;/\sqrt{1-z^{\,2}}\right)$$

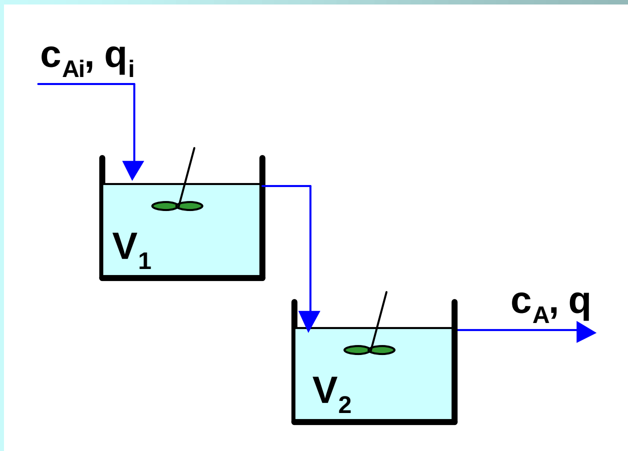
  - **Period of oscillation (P)** $\quad P = 2pt\;/\sqrt{1-z^{\,2}}$



2^{ND} ORDER PROCESSES

- **Two tanks in series**
  - **If $v_1 = v_2$, critically damped.**
  - **Or, overdamped (no oscillation)**

$$\frac{C_A(s)}{C_{Ai}(s)} = \frac{1}{((V_1/q)s+1)((V_2/q)s+1)}$$



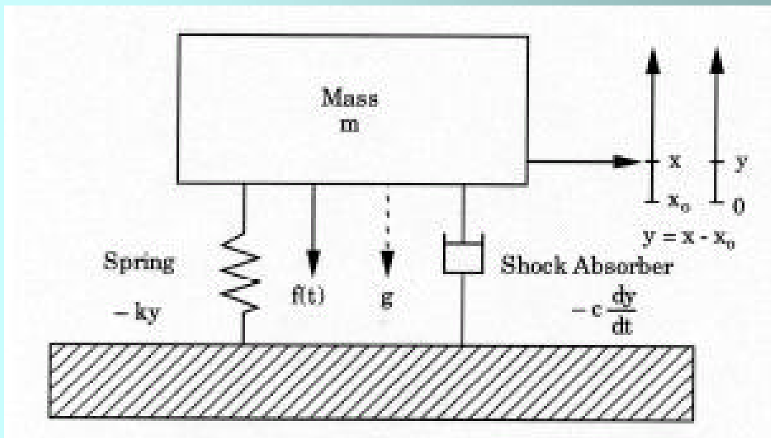
- **Spring-dashpot (shock absorber)**
  - **By force balance**

$$(mg + f(t)) - ky - cv = ma$$

$$my'' = -ky - cy' + (mg + f(t))$$

$$\left(\sqrt{\frac{m}{k}}\right)^2 y'' + 2\sqrt{\frac{c^2}{4mk}}\sqrt{\frac{m}{k}}\,y' + y = \tilde{f}(t)$$

τ (the highlighted $\sqrt{m/k}$) ζ (the highlighted $\sqrt{c^2/4mk}$) (can be <1: underdamped)



Underdamped Processes

- Many examples can be found in mechanical and electrical system.
- Among chemical processes, open-loop underdamped process is quite rare.
- However, when the processes are controlled, the responses are usually underdamped.
- Depending on the controller tuning, the shape of response will be decided.
- Slight overshoot results short rise time and often more desirable.
- Excessive overshoot may results long-lasting oscillation.

POLES AND ZEROS

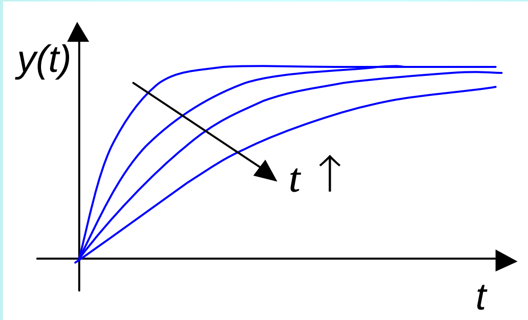
$$G(s) = \frac{N(s)}{D(s)} = \frac{K(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + 1)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1)}$$

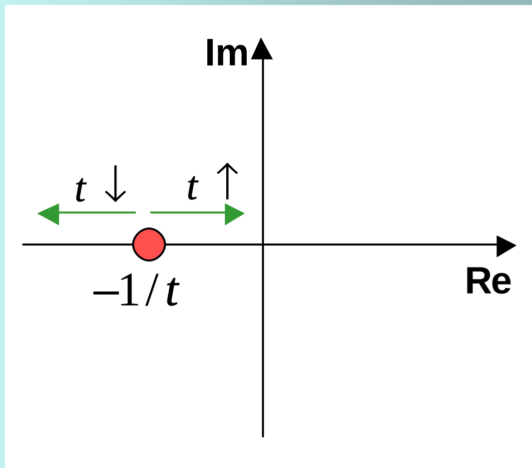
- **Poles ($D(s)$=0)**
  - **Where a transfer function cannot be defined.**
  - **Roots of the denominator of the transfer function**
  - **Modes of the response**
  - **Decide the stability**
- **Zero ($N(s)$=0)**
  - **Where a transfer function becomes zero.**
  - **Roots of the numerator of the transfer function**
  - **Decide weightings for each mode of response**
  - **Decide the size of overshoot or inverse response**
- **They can be real or complex**

- **Real pole from** $(t\,s+1)$

$$s=-\frac{1}{t}$$

  – **Mode:** $e^{-t/t}$





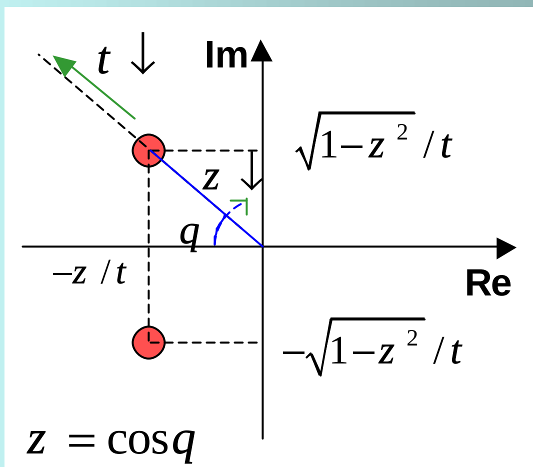
  – **If the pole is at the origin, it becomes “integrating pole.”**
  – **If the pole is in RHP, the response increases exponentially.**

- **Complex pole from** $(t^2 s^2 + 2z\,t\,s+1)$ $(-1 < z < 1)$

$$s=-\frac{z}{t}\pm j\frac{\sqrt{1-z^2}}{t}=-a\pm jb$$

$$|s|=\sqrt{\frac{z^2+1-z^2}{t^2}}=\frac{1}{t}\quad \text{(function of } t \text{ only)}$$

$$\measuredangle s=\pm\tan^{-1}\frac{\sqrt{1-z^2}}{z}\quad \text{(function of } z \text{ only)}$$



– **Modes:** $e^{-at\pm jbt} = e^{-at}(\cos bt \pm j\sin bt)$

$$= e^{-zt/t}\left(\cos\frac{\sqrt{1-z^2}}{t}t \pm j\sin\frac{\sqrt{1-z^2}}{t}t\right)$$

– **Assume t is positive.**

– **If $z<0$, the exponential part will grow as t increases: unstable**

– **If $z>0$, the exponential part will shrink as t increases: stable**

– **If $z=0$, the roots are pure imaginary: sustained oscillation**

- **Effect of zero**

$$G(s) = \frac{N(s)}{(s+p_1)\cdots(s+p_n)} = w_1\frac{1}{(s+p_1)} + \cdots + w_n\frac{1}{(s+p_n)}$$

– **The effects on weighting factors are not obvious, but it is clear that the numerator (zeros) will change the weighting factors.**

EFFECTS OF ZEROS

- **Lead-lag module**

$$G(s)=\frac{N(s)}{D(s)}=\frac{K(t_a s+1)}{(t_1 s+1)}$$

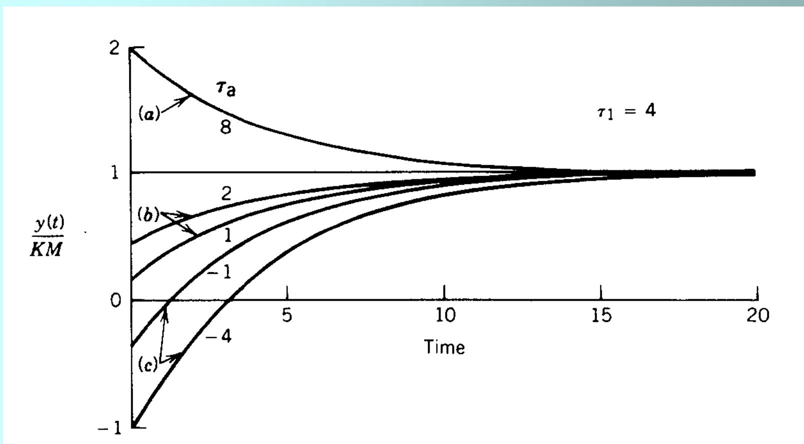
Lead (numerator), Lag (denominator)

– **Depending on the location of zero**

$$Y(s)=\frac{KM(t_a s+1)}{s(t_1 s+1)}=KM\left\{\frac{1}{s}+\frac{t_a-t_1}{t_1 s+1}\right\} \qquad y(t)=KM\left[1-\left(1-\frac{t_a}{t_1}\right)e^{-t/t_1}\right]$$

(a) $t_a > t_1 > 0$

The lead dominates the lag.



(b) $0 \le t_a < t_1$

The lag dominates the lead.

(c) $0 > t_a$

Inverse response

- **Overdamped 2nd order+single zero system**

$$G(s) = \frac{N(s)}{D(s)} = \frac{K(t_a s+1)}{(t_1 s+1)(t_2 s+1)}$$

$$Y(s) = \frac{KM(t_a s+1)}{s(t_1 s+1)(t_2 s+1)} = KM\left\{\frac{1}{s} + \frac{t_1(t_a - t_1)}{t_1 - t_2}\frac{1}{t_1 s+1} + \frac{t_2(t_a - t_2)}{t_2 - t_1}\frac{1}{t_2 s+1}\right\}$$

$$y(t) = KM\left[1 + \frac{t_a - t_1}{t_1 - t_2}e^{-t/t_1} + \frac{t_a - t_2}{t_2 - t_1}e^{-t/t_2}\right]$$

(a) $t_a > t_1 > 0$ (assume $t_1 > t_2$)

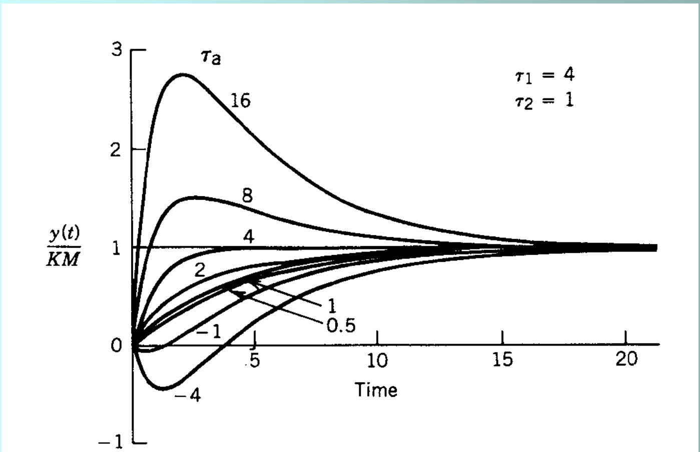
The lead dominates the lags.

(b) $0 < t_a \le t_1$

The lags dominate the lead.

(c) $0 > t_a$

Inverse response

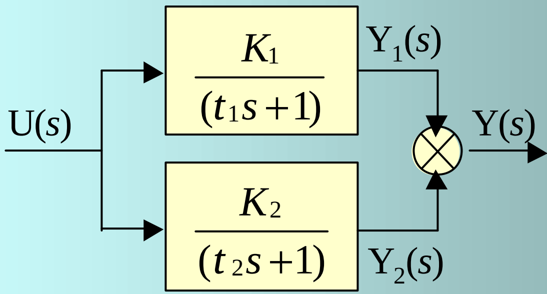


- **Other interpretation**

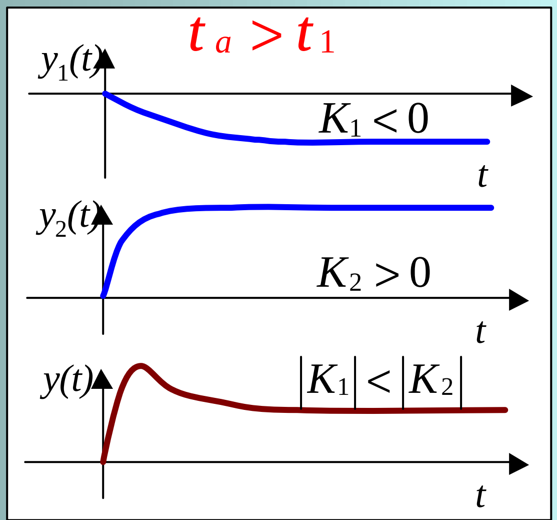
$$G(s)=\frac{K(t_a s+1)}{(t_1 s+1)(t_2 s+1)}=\frac{K_1}{(t_1 s+1)}+\frac{K_2}{(t_2 s+1)}$$

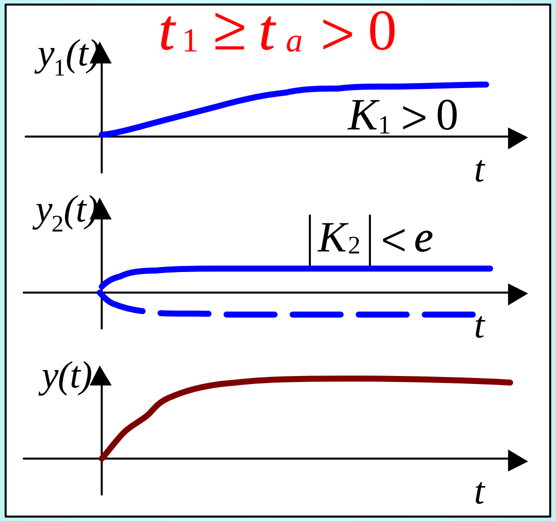
$$K_1=\left.\frac{K(t_a s+1)}{(t_2 s+1)}\right|_{s=-1/t_1}=\frac{K(t_1-t_a)}{(t_1-t_2)}$$

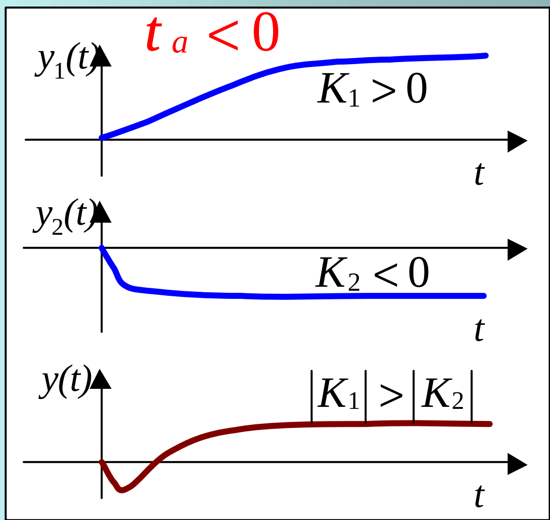
$$K_2=\left.\frac{K(t_a s+1)}{(t_1 s+1)}\right|_{s=-1/t_2}=\frac{K(t_a-t_2)}{(t_1-t_2)}$$



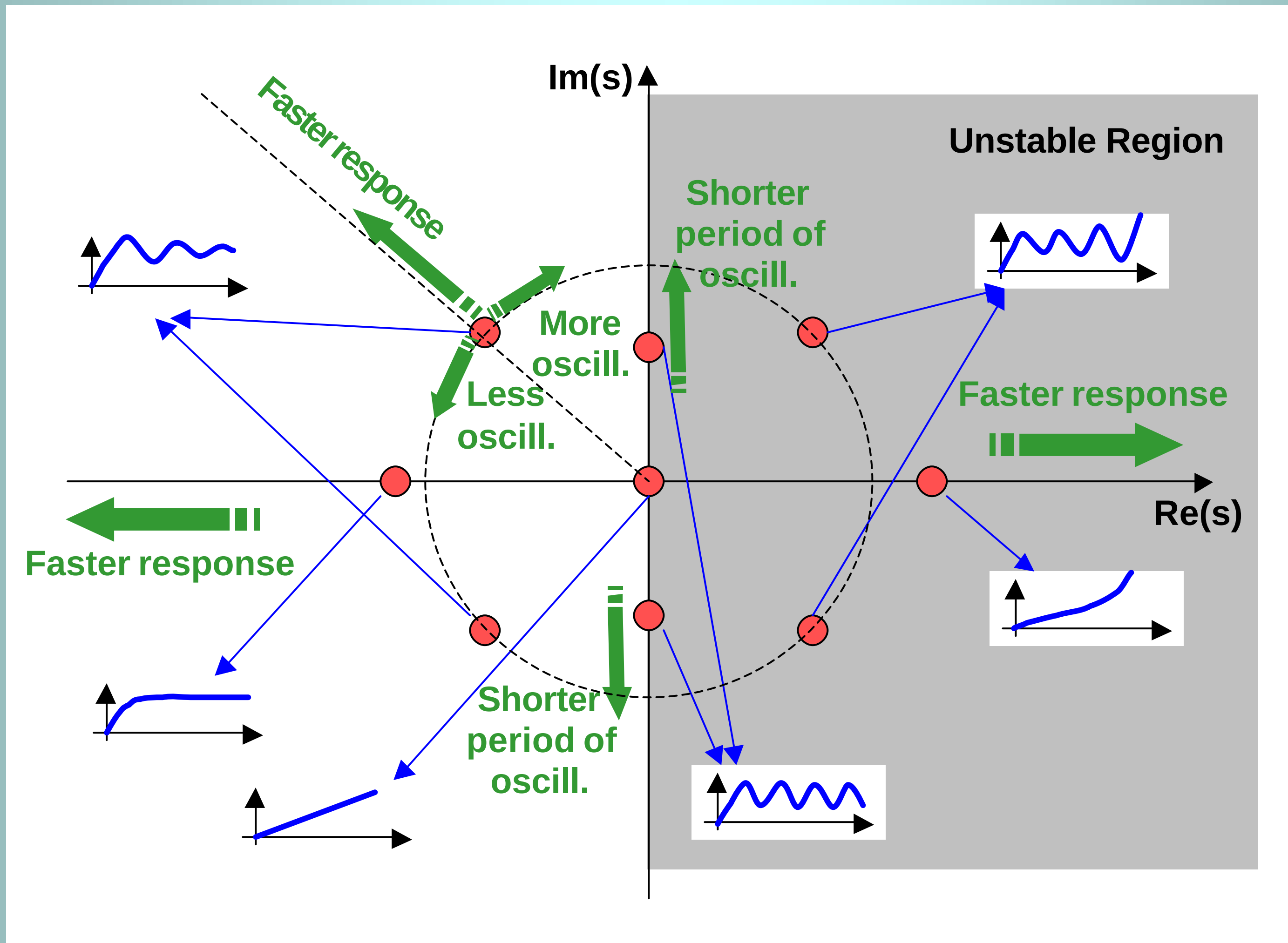
  - **Since $t_1 > t_2$, 1 is slow dynamics and 2 is fast dynamics.**







EFFECTS OF POLE LOCATION



EFFECTS OF ZERO LOCATION