CHE302 LECTURE IX
FREQUENCY RESPONSES

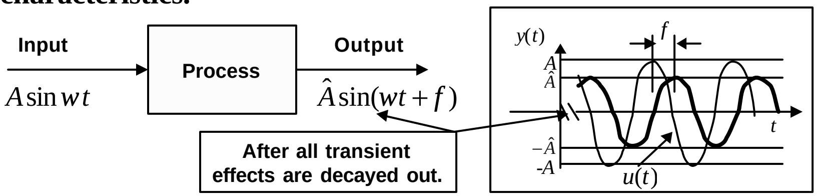
Professor Dae Ryook Yang

Fall 2001

Dept. of Chemical and Biological Engineering
Korea University

DEFINITION OF FREQUENCY RESPONSE

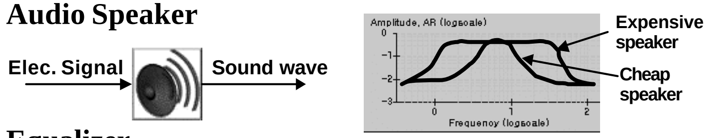
- **For linear system**
  - **"The ultimate output response of a process for a sinusoidal input at a frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics."**



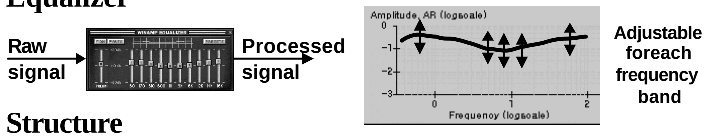
  - **Amplitude ratio (AR): attenuation of amplitude, $\hat{A}/A$**
  - **Phase angle (f): phase shift compared to input**
  - **These two quantities are the function of frequency.**

BENEFITS OF FREQUENCY RESPONSE

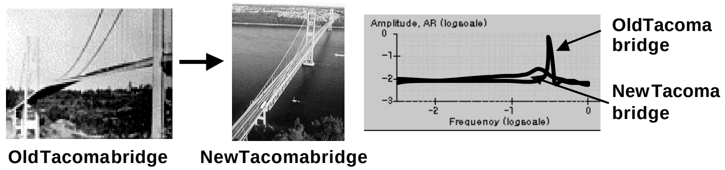
- **Frequency responses are the informative representations of dynamics systems**
  - **Audio Speaker**



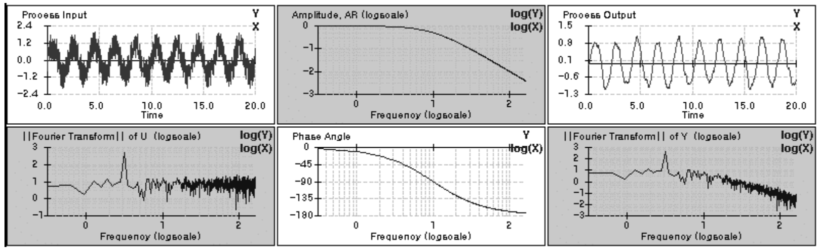
  - **Equalizer**



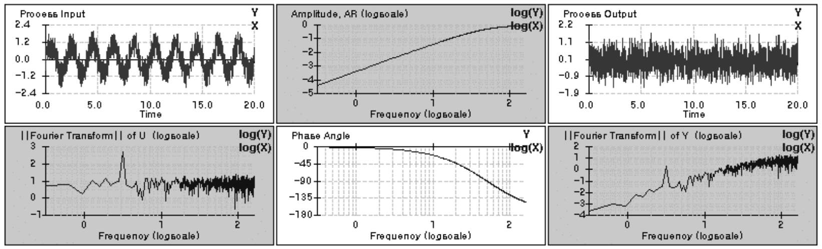
  - **Structure**



  - **Low-pass filter**



  - **High-pass filter**

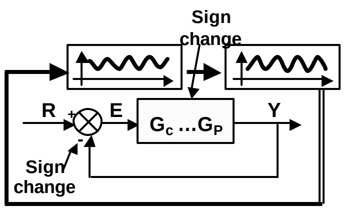


  - **In signal processing field, transfer functions are called "filters".**

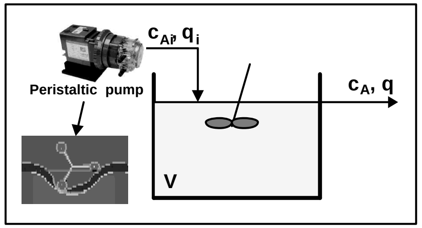
- **Any linear dynamical system is completely defined by its frequency response.**
  - **The AR and phase angle define the system completely.**
  - **Bode diagram**
    - **AR in log-log plot**
    - **Phase angle in log-linear plot**
  - **Via efficient numerical technique (fast Fourier transform, FFT), the output can be calculated for any type of input.**

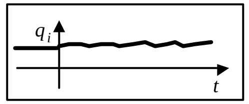
- **Frequency response representation of a system dynamics is very convenient for designing a feedback controller and analyzing a closed-loop system.**
  - **Bode stability**
  - **Gain margin (GM) and phase margin (PM)**

- **Critical frequency**
  - **As controller gain changes, the amplitude ratio (AR) and the phase angle (PA) change.**
  - **The frequency where the PA reaches –180° is called critical frequency (w_c).**
  - **The component of output at the critical frequency will have the exactly same phase as the signal goes through the loop due to comparator (-180 °) and phase shift of the process (-180 °).**
  - **For the open-loop gain at the critical frequency,** $K_{OL}(w_c) = 1$
    - **No change in magnitude**
    - **Continuous cycling**
  - **For** $K_{OL}(w_c) > 1$
    - **Getting bigger in magnitude**
    - **Unstable**
  - **For** $K_{OL}(w_c) < 1$
    - **Getting smaller in magnitude**
    - **Stable**



- **Example**
  - **If a feed is pumped by a peristaltic pump to a CSTR, will the fluctuation of the feed flow appear in the output?**



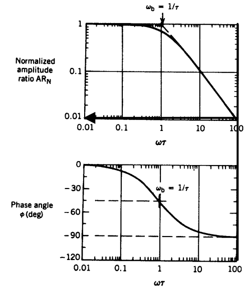


$$V\frac{dc_A}{dt} = q_i c_{Ai} - q c_A \quad (q \approx \text{constant})$$

$$\frac{C_A(s)}{q_i(s)} = \frac{C_{Ai}}{Vs+q} = \frac{C_{Ai}/q}{(V/q)s+1}$$

  - **V=50cm^3, q=90cm^3/min (so is the average of q_i)**
    - **Process time constant=0.555min.**
  - **The rpm of the peristaltic pump is 60rpm.**
    - **Input frequency=180rad/min (3blades)**
  - **The AR=0.01** $(w\tau = 100)$

If the magnitude of fluctuation of q_i is 5% of nominal flow rate, the fluctuation in the output concentration will be about 0.05% which is almost unnoticeable.



OBTAINING FREQUENCY RESPONSE

- **From the transfer function, replace s with jw**

$$G(s) \xrightarrow{s=jw} G(jw)$$

Transfer function Frequency response

  - **For a pole,** $s = a + jw$ **, the response mode is** $e^{(a+jw)t}$.
  - **If the modes are not unstable (**$a \le 0$**) and enough time elapses, the survived modes becomes** e^{jwt}. **(ultimate response)**

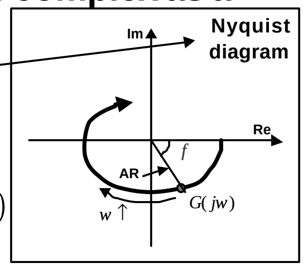
- **The frequency response,** $G(jw)$ **is complex as a function of frequency.**

$$G(jw) = \text{Re}[G(jw)] + j\,\text{Im}[G(jw)]$$

$$AR = |G(jw)| = \sqrt{\text{Re}[G(jw)]^2 + \text{Im}[G(jw)]^2}$$

$$f = \measuredangle G(jw) = \tan^{-1}\left(\text{Im}[G(jw)]/\text{Re}[G(jw)]\right)$$

Bode plot



- **Getting ultimate response**
  - For a sinusoidal forcing function $Y(s)=G(s)\dfrac{Aw}{s^2+w^2}$
  - Assume $G(s)$ has stable poles b_i.

Decayed out at large t

$$Y(s)=G(s)\frac{Aw}{s^2+w^2}=\frac{a_1}{s+b_1}+\cdots+\frac{a_n}{s+b_n}+\frac{Cs+Dw}{s^2+w^2}$$

$$G(jw)Aw=Cjw+Dw\Rightarrow G(jw)=\frac{D}{A}+j\frac{C}{A}=R+jI$$

$$C=IA,\ D=RA\ \Rightarrow y_{ul}=A(I\cos wt+R\sin wt)=\hat{A}\sin(wt+f)$$

$$\therefore AR=\hat{A}/A=\sqrt{R^2+I^2}=|G(jw)|\ \text{and}\ f=\tan^{-1}(I/R)=\measuredangle G(jw)$$

  - Without calculating transient response, the frequency response can be obtained directly from $G(jw)$.
  - Unstable transfer function does not have a frequency response because a sinusoidal input produces an unstable output response.

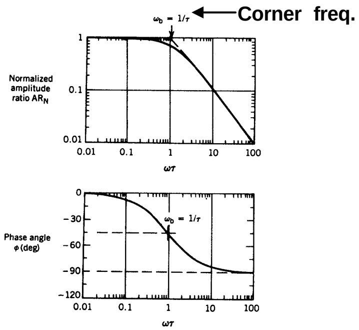
- **First-order process**

$$G(s)=\frac{K}{(ts+1)}$$

$$G(jw)=\frac{K}{(1+jwt)}=\frac{K}{(1+w^2t^2)}(1-jwt)$$

$$AR_N=|G(jw)|=\frac{1}{\sqrt{1+w^2t^2}}$$

$$f=\measuredangle G(jw)=-\tan^{-1}(wt)$$



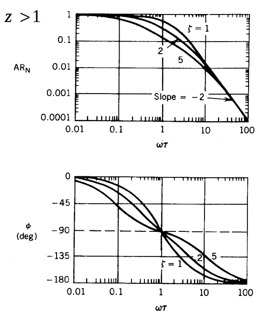
- **Second-order process**

$$G(s)=\frac{K}{(t^2s^2+2zts+1)}$$

$$G(jw)=\frac{K}{(1-t^2w^2)+2jztw}$$

$$AR=|G(jw)|=\frac{K}{\sqrt{(1-w^2t^2)^2+(2zwt)^2}}$$

$$f=\measuredangle G(jw)=\tan^{-1}\frac{\text{Im}(G(jw))}{\text{Re}(G(jw))}=-\tan^{-1}\frac{2zwt}{1-w^2t^2}$$



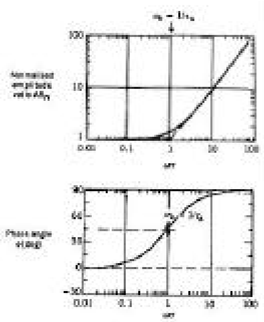
- **Process Zero (lead)**

$$G(s)=t_as+1$$

$$G(jw)=1+jwt_a$$

$$AR_N=|G(jw)|=\sqrt{1+w^2t_a^2}$$

$$f=\measuredangle G(jw)=\tan^{-1}(wt_a)$$



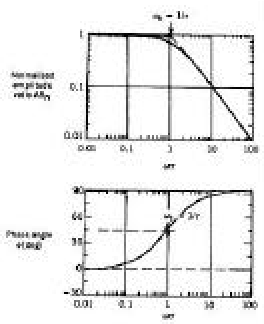
- **Unstable pole**

$$G(s)=\frac{1}{(-ts+1)}$$

$$G(jw)=\frac{1}{1-jtw}=\frac{1}{1+t^2w^2}(1+jtw)$$

$$AR=|G(jw)|=\frac{1}{\sqrt{1+w^2t^2}}$$

$$f=\measuredangle G(jw)=\tan^{-1}\frac{\text{Im}(G(jw))}{\text{Re}(G(jw))}=\tan^{-1}wt$$

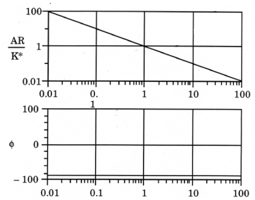


- **Integrating process**

$$G(s)=\frac{1}{As}\qquad G(jw)=\frac{1}{jAw}=-\frac{1}{Aw}j$$

$$AR_N=|G(jw)|=\frac{1}{Aw}$$

$$f=\measuredangle G(jw)=\tan^{-1}(-\frac{1}{0\cdot w})=-\frac{p}{2}$$

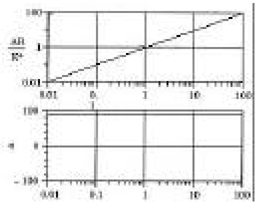


- **Differentiator**

$$G(s)=As\qquad G(jw)=jAw$$

$$AR_N=|G(jw)|=Aw$$

$$f=\measuredangle G(jw)=\tan^{-1}(\frac{1}{0\cdot w})=\frac{p}{2}$$



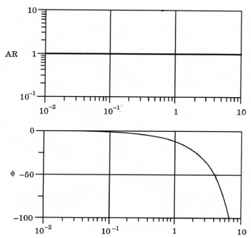
- **Pure delay process**

$$G(s)=e^{-qs}$$

$$G(jw)=e^{-jqw}=\cos qw-j\sin qw$$

$$AR=|G(jw)|=1$$

$$f=\measuredangle G(jw)=-\tan^{-1}\tan qw=-qw$$



SKETCHING BODE PLOT

$$G(s)=\frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots}\qquad G(jw)=\frac{G_a(jw)G_b(jw)G_c(jw)\cdots}{G_1(jw)G_2(jw)G_3(jw)\cdots}$$

$$|G(jw)|=\frac{|G_a(jw)||G_b(jw)||G_c(jw)|\cdots}{|G_1(jw)||G_2(jw)||G_3(jw)|\cdots}$$

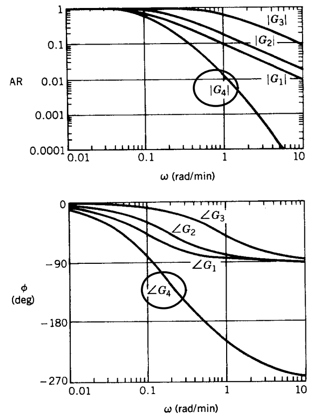
$$\measuredangle G(jw)=\measuredangle G_a(jw)+\measuredangle G_b(jw)+\measuredangle G_c(jw)+\cdots$$
$$-\measuredangle G_1(jw)-\measuredangle G_2(jw)-\measuredangle G_3(jw)-\cdots$$

- **Bodediagram**
  - AR vs. frequency in log-log plot
  - PA vs. frequency in semi-log plot
  - Useful for
    - Analysis of the response characteristics
    - Stability of the closed-loop system only for open-loop stable systems with phase angle curves exhibit a single critical frequency.

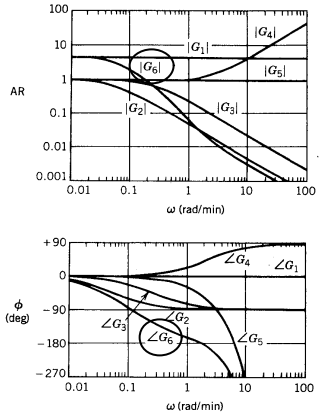
- **Amplitude Ratio on log-log plot**
  - Start from steady-state gain at w =0. If G_{OL} includes either integrator or differentiator it starts at ∞ or 0.
  - Each first-order lag (lead) adds to the slope –1 (+1) starting at the corner frequency.
  - Each integrator (differentiator) adds to the slope –1 (+1) starting at zero frequency.
  - A delays does not contribute to the AR plot.
- **Phase angle on semi-log plot**
  - Start from 0° or -180° at w =0 depending on the sign of steady-state gain.
  - Each first-order lag (lead) adds 0° to phase angle at w=0, adds -90° (+90°) to phase angle at w =∞, and adds -45° (+45°) to phase angle at corner frequency.
  - Each integrator (differentiator) adds -90° (+90°) to the phase angle for all frequency.
  - A delay adds $-qw$ to phase angle depending on the frequency.

Examples

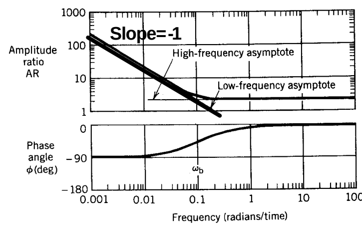
1. $G(s)=\dfrac{K}{(10s+1)(5s+1)(s+1)}$ (denominator factors: G_1, G_2, G_3)



2. $G(s)=\dfrac{5(0.5s+1)e^{-0.5s}}{(20s+1)(4s+1)}$ (numerator: G_1, G_5, G_4; denominator: G_2, G_3)

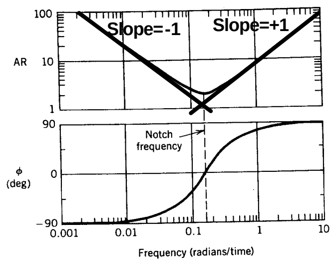


3. PI: $G(s)=K_C\left(1+\dfrac{1}{t_I s}\right)$



$w_b=1/t_I$ at $f=-45°$

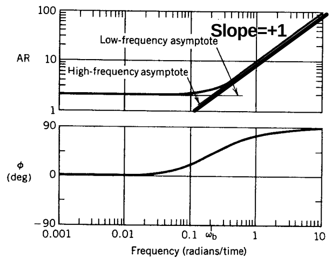
5. PID: $G(s)=K_C\left(1+\dfrac{1}{t_I s}+t_D s\right)$



$w_{Notch}=1/\sqrt{t_I t_D}$ at $f=0°$

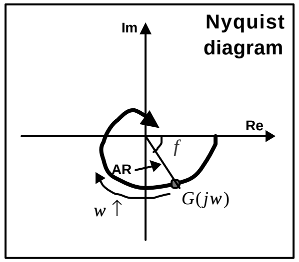
4. PD: $G(s)=K_C(1+t_D s)$

$w_b=1/t_D$ at $f=-45°$



NYQUIST DIAGRAM

- **Alternativerepresentationoffrequencyresponse**
- **Polar plot of** $G(jw)$ **(**w **is implicit)**



$$G(jw) = \text{Re}[G(jw)] + j\,\text{Im}[G(jw)]$$

- **Compact (one plot)**
- **Wider applicability of stability analysis than Bode plot**
- **High frequency characteristics will be shrunk near the origin.**
  - **Inverse Nyquist diagram: polar plot of** $1/G(jw)$
- **Combination of different transfer function components is not easy as with Nyquist diagram as with Bode plot.**