Chap. 9. Vector Integral Calculus. Integral Theorems

- Green, Gauss, Stokes theorems …

9.1. Line Integrals

- Line integral (or curve integral): $\int_a^b f(x)dx$

Definition and Evaluation of Line Integrals

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} dt$$

(curve C = *path of integration*: $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$)

(*every path of integration of a line integral* → *piecewise smooth*)

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z') dt$$

For closed curve: $\int_C \Rightarrow \oint_C$

ex.) Work done by a force $\underline{F}$ in a displacement along C

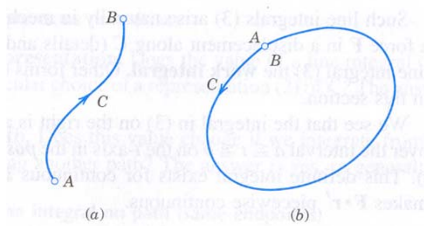

Fig. 200. Oriented curve

Ex. 1) Evaluation of a line integral in the plane

$$\underline{F}(\underline{r}) = -y\underline{i} - xy\underline{j},\ \ \underline{r}(t) = \cos t\underline{i} + \sin t\underline{j}$$

$$\underline{F}(\underline{r}(t)) = -\sin t\underline{i} - \cos t \sin t\underline{j},\ \ \underline{r}'(t) = -\sin t\underline{i} + \cos t\underline{j}$$

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} dt = \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t) dt$$

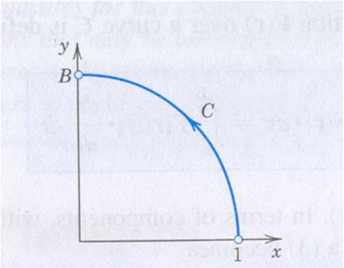

Fig. 201. Example 1

Ex. 2) Line integral in space

$$\underline{F}(\underline{r}) = z\underline{i} + x\underline{j} + y\underline{k},\ \ \underline{r}(t) = \cos t\underline{i} + \sin t\underline{j} + 3t\underline{k},\ \underline{r}'(t) = -\sin t\underline{i} + \cos t\underline{j} + 3\underline{k}$$

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3\sin t) dt$$

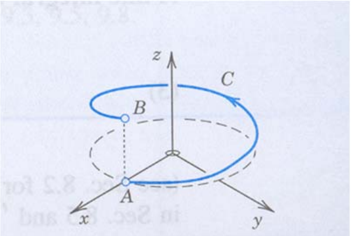

Fig. 202. Example 2

- Choice of representation ?
- Choice of path ?

Ex. 3) Dependence of a line integral on path (from A(0,0,0) to B(1,1,1))

$$\underline{F}(\underline{r}) = 5z\underline{i} + xy\underline{j} + x^2 z\underline{k},\ \ \underline{r}_1(t) = t\underline{i} + t\underline{j} + t\underline{k}\ (0 \le t \le 1),\ \ \underline{r}_2(t) = t\underline{i} + t\underline{j} + t^2\underline{k}\ (0 \le t \le 1)$$

(a) $\int_{C_1} \underline{F}(\underline{r}) \cdot d\underline{r} = \int_0^1 (5t + t^2 + t^3) dt = \frac{37}{12}$

(b) $\int_{C_2} \underline{F}(\underline{r}) \cdot d\underline{r} = \int_0^1 (5t^2 + t^2 + 2t^5) dt = \frac{28}{12}$

Line integral depends on
- ***F***
- ***Endpoints A, B of the path***
- ***path***

- Conditions for guaranteeing independence of the path: see *Section 9.2*

Motivation of the Line Integral: Work Done by a Force

$$\Delta W_m = \underline{F}(\underline{r}(t_m)) \cdot [\underline{r}(t_{m+1}) - \underline{r}(t_m)] \approx \underline{F}(\underline{r}(t_m)) \cdot \underline{r}'(t_m)\Delta t_m$$

Sum of these n works: $W_n = \Delta W_0 + \cdots + \Delta W_{n-1}$, n→∞: $\Rightarrow$ line integral

Ex. 5) Work done equals the gain in kinetic energy

$$W = \int_C \underline{F} \cdot d\underline{r} = \int_a^b \underline{F}(\underline{r}(t)) \cdot \frac{d\underline{r}}{dt} dt = \int_a^b \underline{F}(\underline{r}(t)) \cdot \underline{v}(t)\, dt$$

$$\underline{F} = m\underline{r}''(t) = m\underline{v}'(t)$$

$$W = \int_a^b m\underline{v}' \cdot \underline{v}\, dt = \int_a^b m\left(\frac{\underline{v} \cdot \underline{v}}{2}\right)' dt = \frac{m}{2}|\underline{v}|^2 \Bigg|_{t=a}^{t=b}$$

Other Forms of Line Integrals

Special cases of line integral when F=$F_1\underline{i}$ or $F_2\underline{j}$ or $F_3\underline{k}$ → $\int_C F_1 dx,\ \int_C F_2 dy,\ \int_C F_3 dz$

$$\int_C f(\underline{r})dt = \int_a^b f(\underline{r}(t))dt \quad (\underline{F} = F_1\underline{i} \Rightarrow f = F_1 x')$$

Ex. 6)

General Properties of the Line Integral

$$\int_C k\underline{F}\cdot d\underline{r} = k\int_C \underline{F}\cdot d\underline{r}$$

$$\int_C \left(\underline{F}+\underline{G}\right)\cdot d\underline{r} = \int_C \underline{F}\cdot d\underline{r} + \int_C \underline{G}\cdot d\underline{r}$$

$$\int_C \underline{F}\cdot d\underline{r} = \int_{C_1} \underline{F}\cdot d\underline{r} + \int_{C_2} \underline{F}\cdot d\underline{r}$$

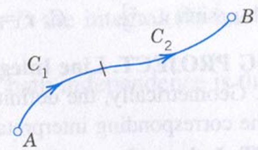

Fig. 204. Formula (8c)

Theorem 1: Direction-preserving transformations of parameter

Any representations of C that give the same positive direction on C also yield the same value of the line integral.

Proof:
$$\int_C \underline{F}(\underline{r}^*)\cdot d\underline{r}^* = \int_{a^*}^{b^*}\left[\underline{F}(\underline{r}^*(t^*))\cdot\frac{d\underline{r}^*}{dt^*}\right]dt^* = \int_{a^*}^{b^*}\underline{F}(\underline{r}(\phi(t^*)))\cdot\frac{d\underline{r}}{dt}\frac{dt}{dt^*}dt^*$$

$$= \int_a^b \underline{F}(\underline{r}(t))\cdot\frac{d\underline{r}}{dt}dt = \int_C \underline{F}(\underline{r})\cdot d\underline{r}$$