Polymath example: Ordinary Differential Equations

Problem

Laminar flow in a horizontal pipe.

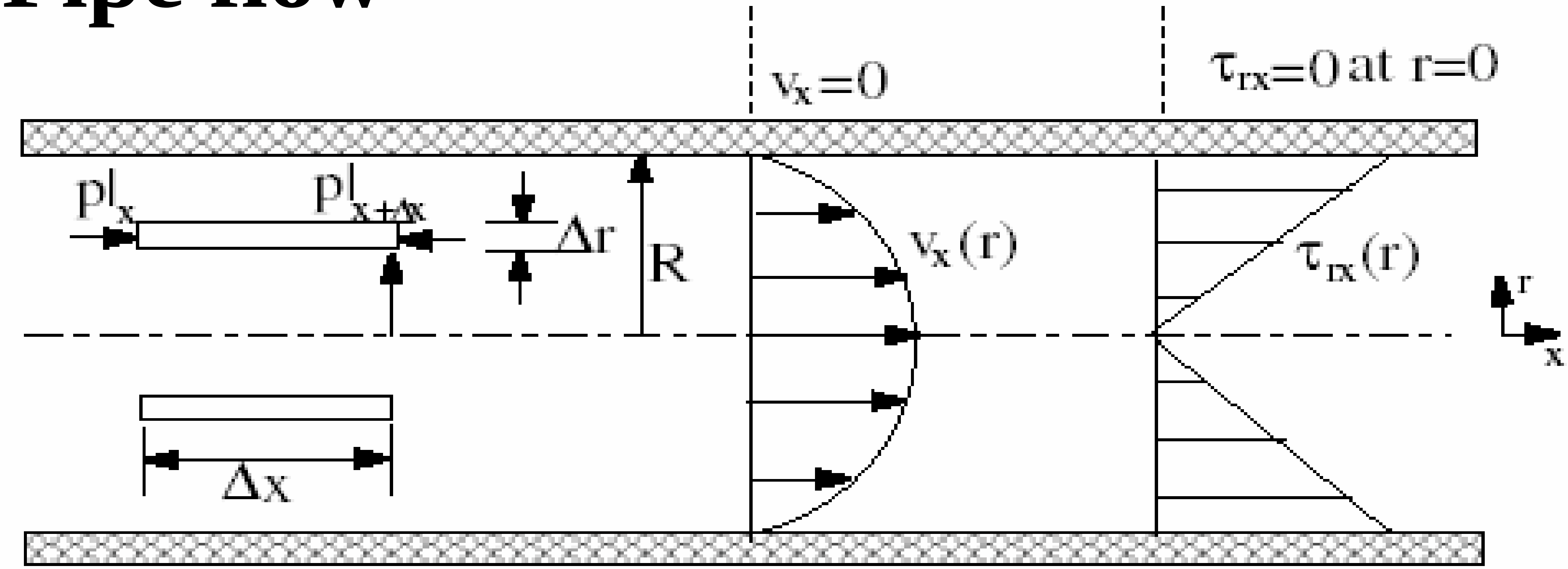
Concepts

Solution of momentum balance to obtain share stress and velocity profiles for a Newtonian fluid in a horizontal pipe : comparison of numerical and analytical solutions.

Numerical methods

Solution of Simultaneous first order ordinary differential equations employing a shooting technique to converge on the desired boundary conditions, and avoidance of division by zero in calculating expressions.

Pipe flow



A shell momentum balance.

$$\frac{d}{dr}(r\tau_{rx}) = \left(\frac{\Delta p}{L}\right) r \qquad (5\text{-}1)$$

Shear stress of Newtonian fluid

$$\tau_{rx} = -\mu \frac{dv_x}{dr} \qquad (5\text{-}2)$$

The boundary conditions for Eq. (5-1) & (5-2)

$$\tau_{rx} = 0 \quad \text{at } r = 0 \tag{5.3}$$

$$v_x = 0 \quad \text{at } r = R \tag{5.4}$$

The analytical solution with the two boundary conditions yields

$$\tau_{rx} = \left(\frac{\Delta p}{2L} r\right) \tag{5.5}$$

$$v_x = \frac{\Delta p}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right] \tag{5.6}$$

Calculation of average

$$v_{x,\mathrm{av}} = \frac{1}{\pi R^2} \int_0^R v_x 2\pi r dr \tag{5.7}$$

Analytical solution

$$v_{x,\mathrm{av}} = \frac{(p_0 - p_L) R^2}{8\mu L} = \frac{(p_0 - p_L) D^2}{32\mu L} \tag{5.8}$$

Problem statements

(a) Numerically solve Eqs. (5-1) and (5-2) with the boundary conditions Eqs. (5-3) and (5-4) for water at 25℃ with μ = .94×10-4kg/m·s, Δp= 500Pa, L=10m, and R=0.009295m.

This solution should utilize an ODE solver with a shooting technique and should employ some techniques for converging on the boundary condition given by Eq.(5-4).

Solution

Rearrange of Eq.(5-2)

$$\frac{dv_x}{dr} = -\frac{\tau_{rx}}{\mu} \qquad (5.9)$$

$$\frac{d}{dr}(r\tau_{rx}) = \left(\frac{\Delta p}{L}\right) r \qquad (5.1)$$

$$\tau_{rx} = 0 \quad \text{at } r = 0 \qquad (5.3)$$

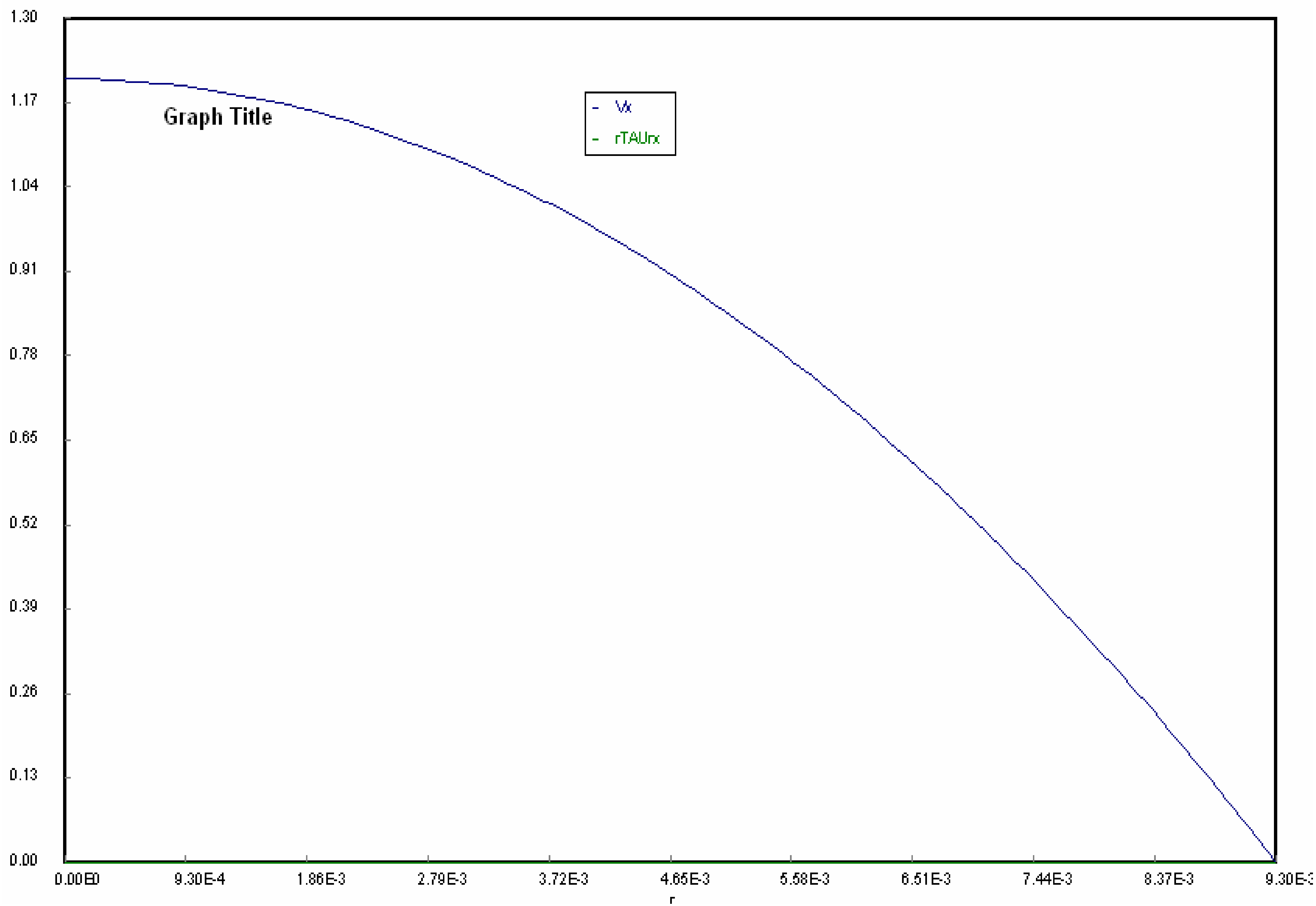
$$v_x = 0 \quad \text{at } r = R \qquad (5.4)$$

Program

```
d(Vx)/d(r) = -TAUrx/mu
d(rTAUrx)/d(r) = deltaP*r/L
deltaP = 500
L = 10
TAUrx = if(r>0)then(rTAUrx/r)else(0)
mu = 8.937e-4
R = .009295
err = Vx-0
```

Variable values

Variable	initial value	minimal value	maximal value	final value
r	0	0	0.009295	0.009295
Vx	1.20842	2.396E-06	1.20842	2.396E-06
rTAUrx	0	0	0.0021599	0.0021599
deltaP	500	500	500	500
L	10	10	10	10
TAUrx	0	0	0.232375	0.232375
mu	8.937E-04	8.937E-04	8.937E-04	8.937E-04
R	0.009295	0.009295	0.009295	0.009295
err	1.20842	2.396E-06	1.20842	2.396E-06


Graph Title
Vx
rTAUrx
1.30
1.17
1.04
0.91
0.78
0.65
0.52
0.39
0.26
0.13
0.00
0.00E0
9.30E-4
1.86E-3
2.79E-3
3.72E-3
4.65E-3
5.58E-3
6.51E-3
7.44E-3
8.37E-3
9.30E-3
r

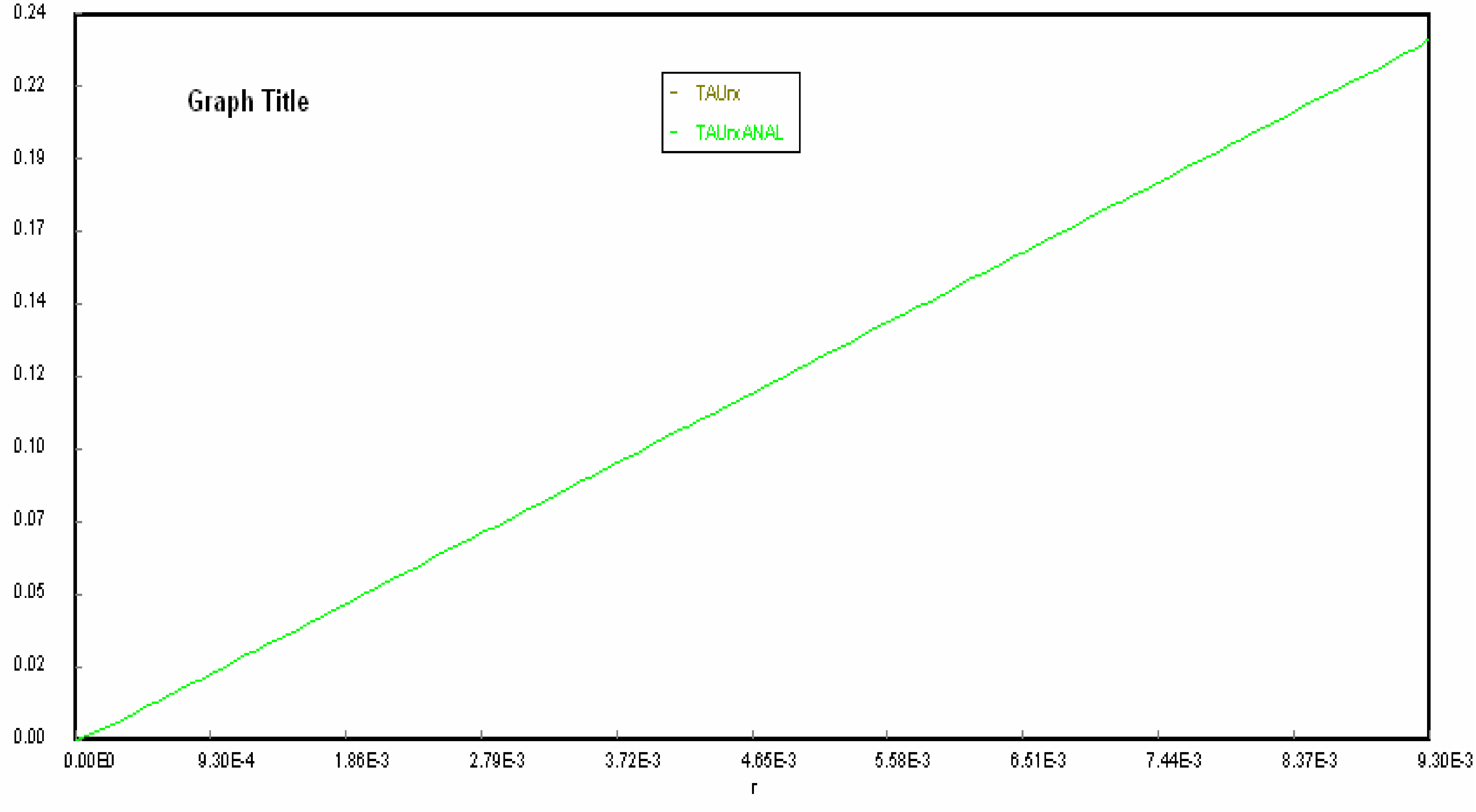
Comparison

(b) Compare the calculated shear stress and velocity profiles with the analytical solutions given by Equations (5-5) and (5-6).

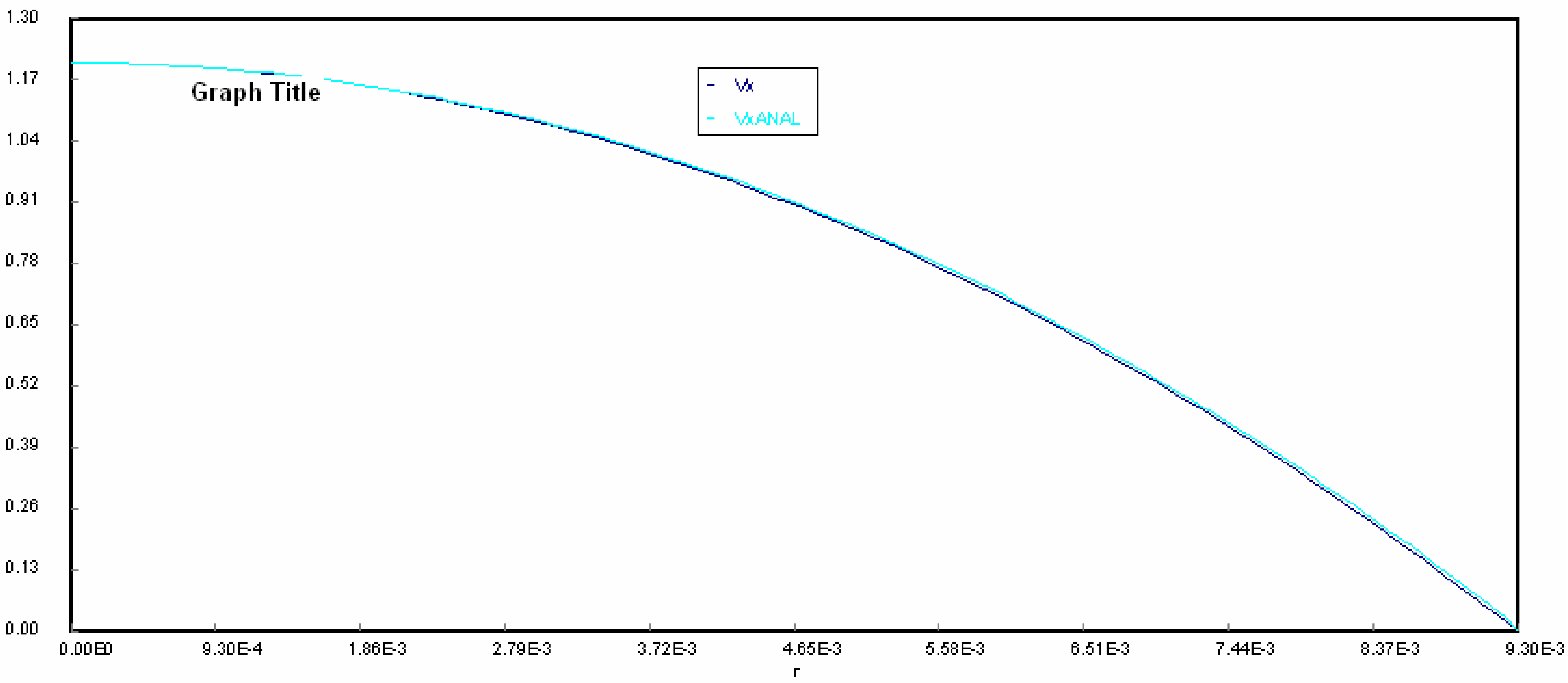
Program

```
d(Vx)/d(r) = -TAUrx/mu
d(rTAUrx)/d(r) = deltaP*r/L
deltaP = 500
L = 10
TAUrx = if(r>0)then(rTAUrx/r)else(0)
mu = 8.937e-4
R = .009295
err = Vx-0
TAUrxANAL = (deltaP/(2*L))*r
VxANAL = (deltaP*R^2/(4*mu*L))*(1-(r/R)^2)
```

Comparison between calculated and analytical shear stress.



Comparison with calculated and analytical velocity profiles.

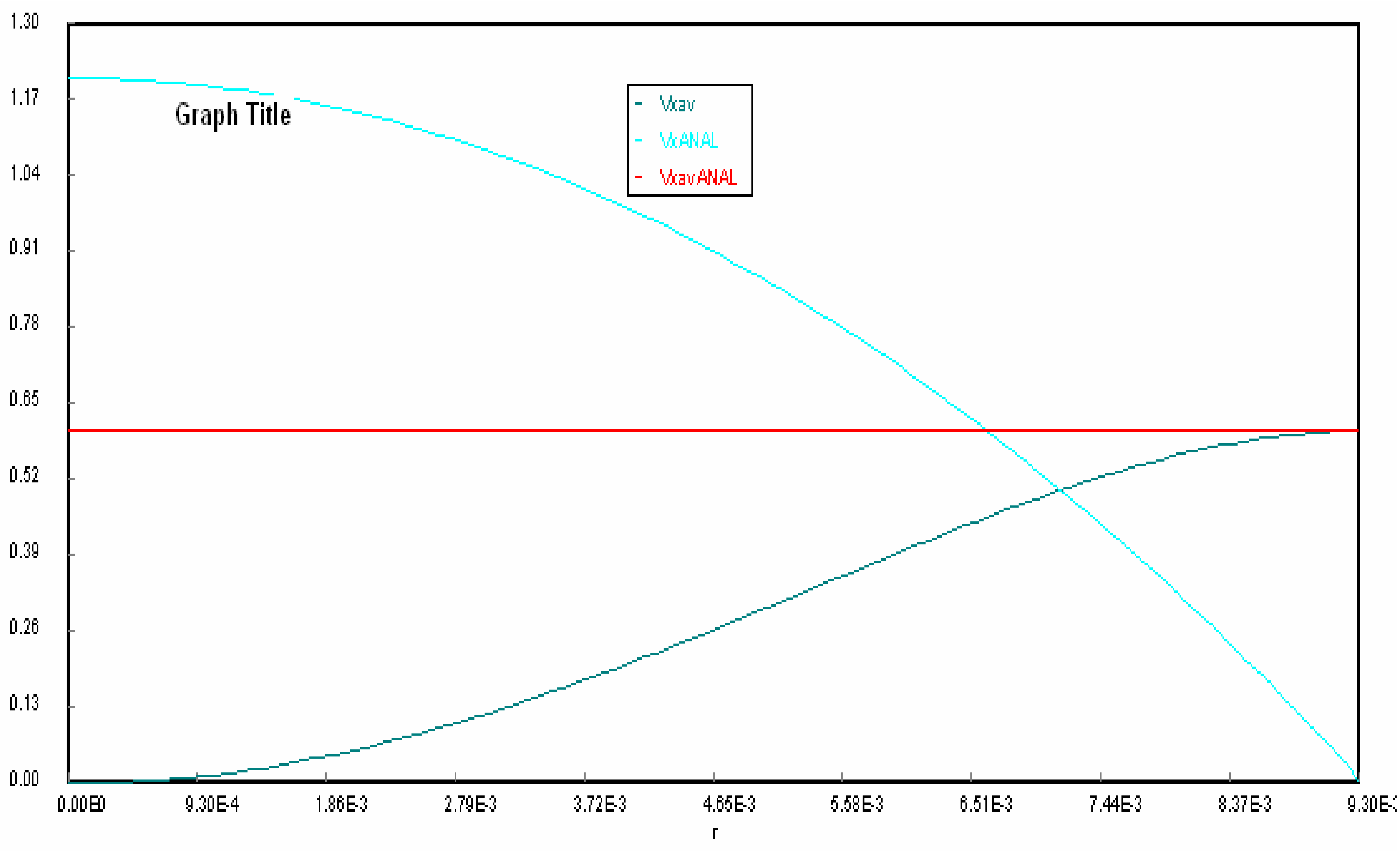


(c) Modify your solution to part (a) to include calculation of the average velocity given by Equation (5-7) and compare your solution with the analytical solution of Equation (5-8).

Rearrange Eq. (5-7)

$$v_{x,av} = \frac{1}{\pi R^2} \int_0^R v_x 2\pi r dr = \frac{v_x 2r}{R^2}$$

```
d(Vx)/d(r) = -TAUrx/mu
d(rTAUrx)/d(r) = deltaP*r/L
d(Vxav)/d(r) = Vx*2*r/R^2
deltaP = 500
L = 10
TAUrx = if(r>0)then(rTAUrx/r)else(0)
mu = 8.937e-4
R = .009295
err = Vx-0
TAUrxANAL = (deltaP/(2*L))*r
VxANAL = (deltaP*R^2/(4*mu*L))*(1-(r/R)^2)
VxavANAL = deltaP*R^2/(8*mu*L)
```



Graph Title
Vxav
VxANAL
VxavANAL
1.30
1.17
1.04
0.91
0.78
0.65
0.52
0.39
0.26
0.13
0.00
0.00E0
9.30E-4
1.86E-3
2.79E-3
3.72E-3
4.65E-3
5.58E-3
6.51E-3
7.44E-3
8.37E-3
r