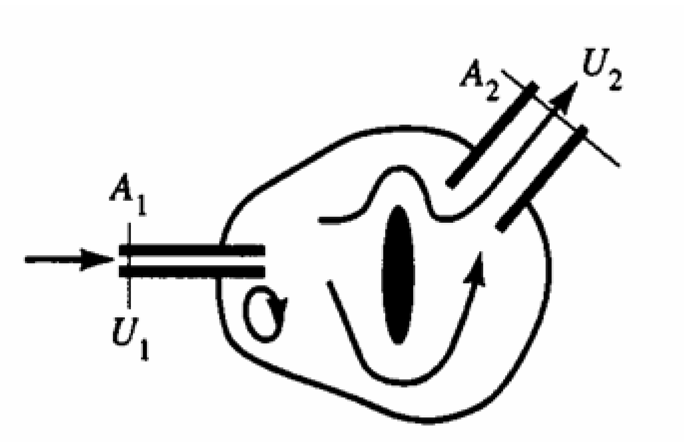
Chapter 11

Macroscopic balances

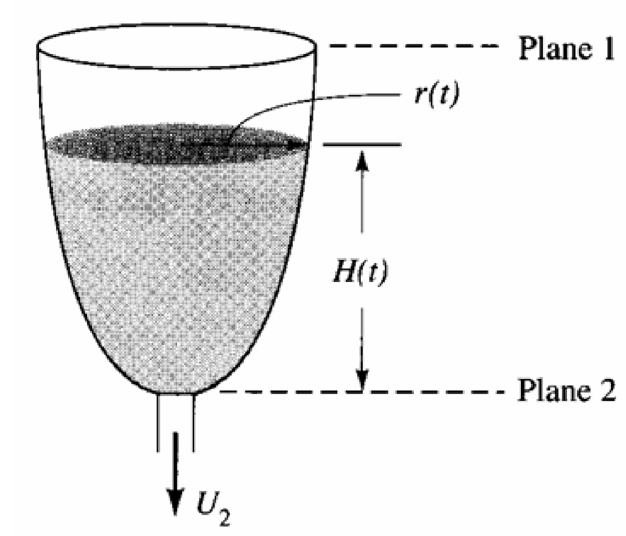
Mass balance



$$\begin{bmatrix}\text{Time rate of change of mass} \\ \text{within the system}\end{bmatrix} = \text{rate of flow of mass in} - \text{rate of flow of mass out}$$

$$\frac{dM}{dt} = [\rho UA]_{\text{in}} - [\rho UA]_{\text{out}} = \rho_1 U_1 A_1 - \rho_2 U_2 A_2$$

Egyptian water clock



Designed such that the water level is a linear function of time

$$\frac{dM}{dt} = \rho_1 U_1 A_1 - \rho_2 U_2 A_2$$

No flow across the plane 1

$$\frac{dM}{dt} = \rho \pi r^2 \frac{dH}{dt} \qquad \rho \pi r^2 \frac{dH}{dt} = \frac{dM}{dt} = -\rho_2 U_2 A_2 = -\rho \sqrt{2gH(t)} A_2$$

Assume Torricelli's law

$$U_2 = \sqrt{2gH(t)}$$

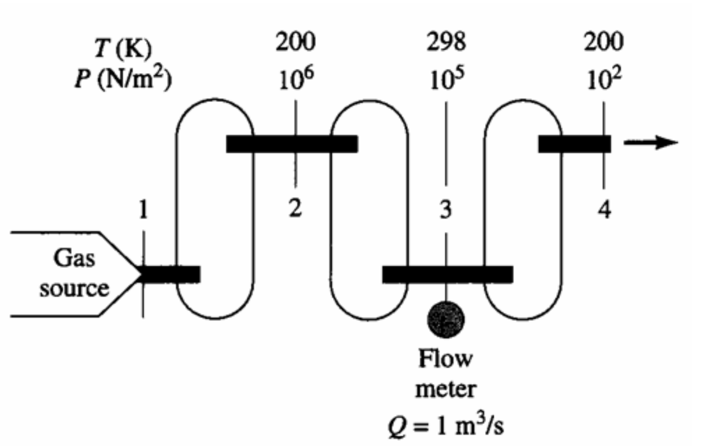
By design $-\frac{dH}{dt} = \dot{H}_o$

$$\frac{r}{A_2^{1/2}} = \frac{(2g)^{1/4}}{(\pi \dot{H}_o)^{1/2}} H^{1/4} = \alpha H^{1/4}$$

Design parameter

Not valid close to bottom

Flow rate of a gas



Between plane 2 and 3

$$\rho_3 U_3 A_3 = \rho_3 Q_3 = \rho_2 U_2 A_2 = \rho_2 Q_2$$

$$Q_2 = \frac{\rho_3}{\rho_2} Q_3 \qquad \frac{\rho_3}{\rho_2} = \frac{P_3}{T_3}\frac{T_2}{P_2}$$

For ideal gas

$$Q_2 = \frac{P_3}{P_2}\frac{T_2}{T_3} Q_3 = \frac{10^5}{10^6}\frac{200}{298} \times 1 = 0.067 \text{ m}^3/\text{s}$$

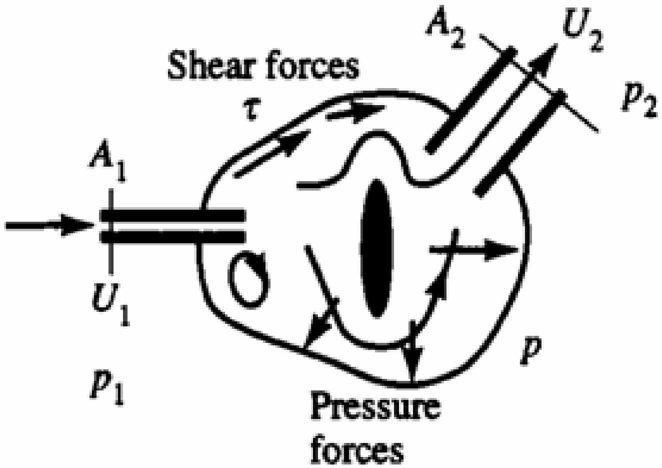
Between plane 3 and 4

$$Q_4 = \frac{P_3}{P_4}\frac{T_4}{T_3} Q_3 = \frac{10^5}{10^2}\frac{200}{298} \times 1 = 670 \text{ m}^3/\text{s}$$

Momentum balance

$$\langle U_1^2 \rangle \equiv \frac{\int u_1^2 \, d\mathbf{A}_1}{\mathbf{A}_1}$$

$$\langle U_1 \rangle \equiv \frac{\int u_1 \, d\mathbf{A}_1}{\mathbf{A}_1} \neq \sqrt{\langle U_1^2 \rangle}$$



rate of change of momentum of the fluid within the system	= net rate of flow of momentum across the surfaces of the system	+ sum of the forces exerted *on the fluid by the boundaries* of the system

+ body force

$$\frac{\partial}{\partial t}\int \rho \mathbf{u} \, dV = \rho_1 \langle U_1^2 \rangle \mathbf{A}_1 - \rho_2 \langle U_2^2 \rangle \mathbf{A}_2 + P_1\mathbf{A}_1 - P_2\mathbf{A}_2 - \mathbf{F} + M\mathbf{g}$$

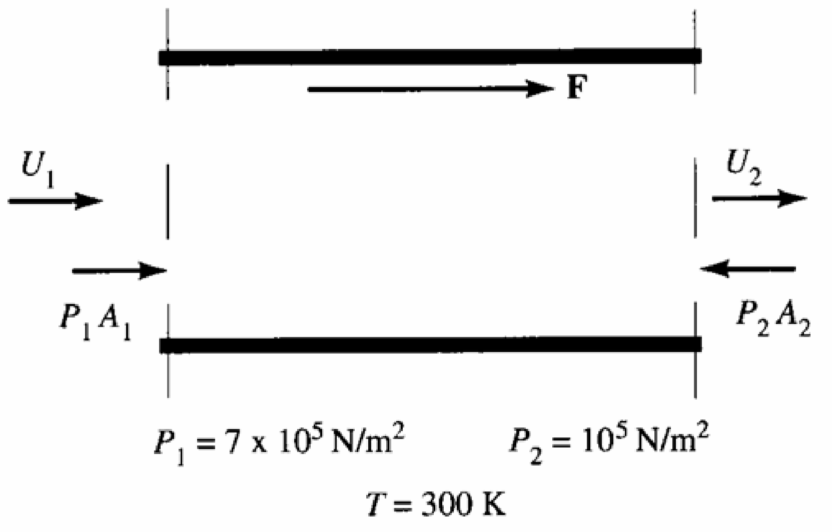
Acting on the open ends of the system

From viscous shear effect and pressure along the surfaces

+ is exerted by the fluid on the boundaries

Guage pressures

Axial force exerted on the walls of a straight pipe



Inside diamter 0.1m
efflux velocity 30m/s
turbulent flow

-> frictional force exerted on the inside surface of the pipe

$$\rho_1 = \frac{PM_w}{R_G T} = \frac{7 \times 10^5 \times 29}{8314 \times 300} = 8.1 \text{ kg/m}^3$$

For ideal gas

Mass balance $\rho_1 \langle U_1 \rangle A_1 = \rho_2 \langle U_2 \rangle A_2$ $\quad \langle U_1 \rangle = \frac{\rho_2}{\rho_1} \langle U_2 \rangle = \frac{1.16}{8.1} \times 30 = 4.3 \text{m/s}$

Momentum balance $\frac{\partial}{\partial t} \int \rho \mathbf{u} \, dV = \rho_1 \langle U_1^2 \rangle \mathbf{A}_1 - \rho_2 \langle U_2^2 \rangle \mathbf{A}_2 + P_1 \mathbf{A}_1 - P_2 \mathbf{A}_2 - \mathbf{F} + M\mathbf{g}$

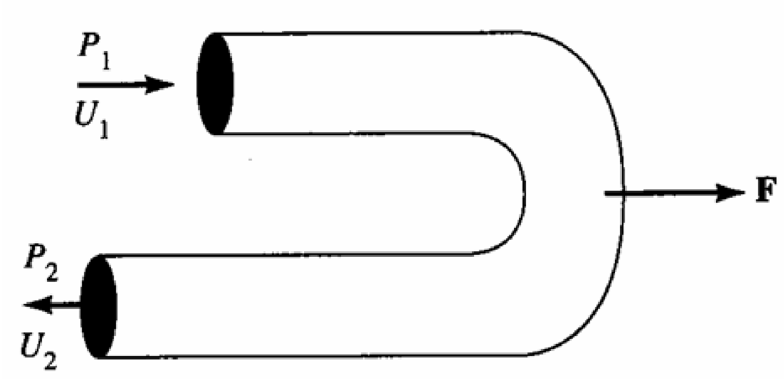
$\langle U_1^2 \rangle = \langle U_1 \rangle^2$ Turbulent flow -> flat velocity profile

$$\mathbf{F} = 8.1(4.3)^2\left[\frac{\pi}{4}(0.1)^2\right] - 1.16(30)^2\left[\frac{\pi}{4}(0.1)^2\right] + [(7\times10^5) - 10^5] - \left[\frac{\pi}{4}(0.1)^2\right]$$
$$- (10^5 - 10^5)\left[\frac{\pi}{4}(0.1)^2\right]$$
$$= 1.18 - 8.2 + \underline{4715} - 0 = 4708\text{N}$$

Major contribution

Equivalent to hanging 450kg on the vertical end of the pipe

Flow of water through a *U*-bend in a pipe



Pipe diameter D=0.1m
$\langle U_1 \rangle$=20m/s
P_1=2x10^{5}Pa (gage)
P_2=1.6x10^{5}Pa (gage)

$$\frac{\partial}{\partial t}\int \rho \mathbf{u}\, dV = \rho_1\langle U_1^2\rangle \mathbf{A}_1 - \rho_2\langle U_2^2\rangle \mathbf{A}_2 + P_1\mathbf{A}_1 - P_2\mathbf{A}_2 - \mathbf{F} + M\mathbf{g}$$

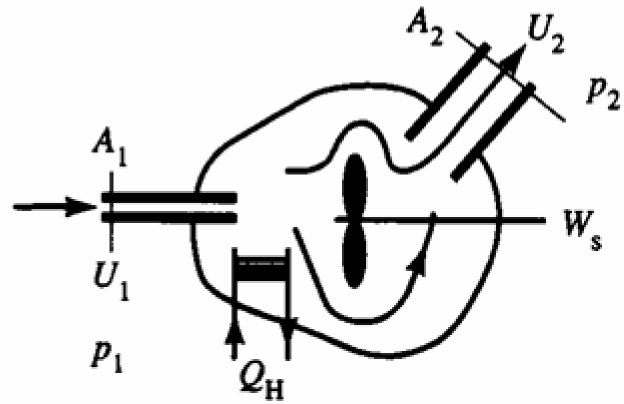
$$\text{F} = 2\left[1000\,\text{kg/m}^3 \times (20\,\text{m/s})^2 \times \frac{\pi}{4}(0.1\text{m})^2\right]$$

$$+\,2\times10^5\,\textbf{Pa}\times\frac{\pi}{4}(0.1\text{m})^2 + 1.6\times10^5\,\textbf{Pa}$$

$$\times\frac{\pi}{4}(0.1\text{m})^2 = 6283 + 2827 = 9110\ \text{N}$$

Major contribution

Energy balance



$$\frac{\text{energy}}{\text{fluid mass}} \equiv \mathsf{E} = \varepsilon + \frac{U^2}{2} + gh$$

The first law of thermodynamics for a flow system

$$\frac{d}{dt}\int \rho \mathsf{E}\, d\mathsf{V} = \langle \rho U A \mathsf{E} \rangle_1 - \langle \rho U A \mathsf{E} \rangle_2 + \underline{Q_H} + \underline{W_s} + \underline{W_f}$$

Heat transfer across the boundary

Shaft work

Flow work

Positive when work is done on the system by the surroundings

Steady state

$$\frac{d}{dt}\int \rho \mathsf{E}\, d\mathsf{V} = \langle \rho U A \mathsf{E} + pUA \rangle_1 - \langle \rho U A \mathsf{E} + pUA \rangle_2 + Q_H + W_s$$

$$0 = \Delta \langle \rho U A \mathsf{E} + pUA \rangle - Q_H - W_s$$

$$0 = \Delta \left\langle \varepsilon + \frac{U^2}{2} + gh + \frac{p}{\rho} \right\rangle - \hat{Q}_H - \hat{W}_s$$

Per mass flow rate $w = \rho U A$

$$0 = d\left\langle \varepsilon + \frac{U^2}{2} + gh + \frac{p}{\rho} \right\rangle - d\hat{Q}_H - d\hat{W}_s$$

$$d\varepsilon = T\,dS - pd\left(\frac{1}{\rho}\right) \qquad T\,dS - dQ_H + \frac{dU^2}{2} + g\,dh + \frac{dp}{\rho} = d\hat{W}_s$$

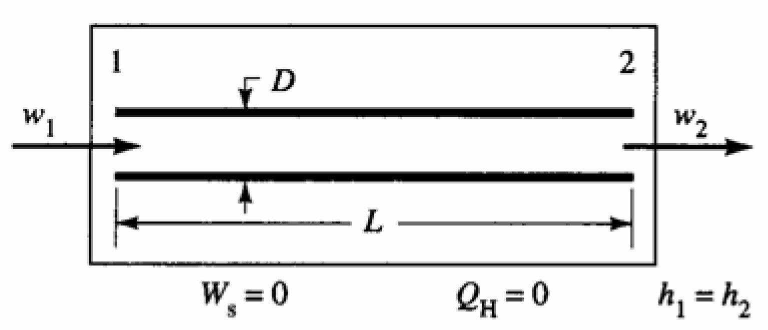
Viscous loss $T\,dS - dQ_H \equiv d\hat{E}_v \geq 0$

$$\frac{dU^2}{2} + g\,dh + \frac{dp}{\rho} = d\hat{W}_s - d\hat{E}_v$$

$$\boxed{\Delta \frac{U^2}{2} + g\Delta h + \int_{p_1}^{p_2} \frac{dp}{\rho} = \hat{W}_s - \hat{E}_v}$$

Bernoulli equation (constant density)

Turbulent flow through a long straight pipe



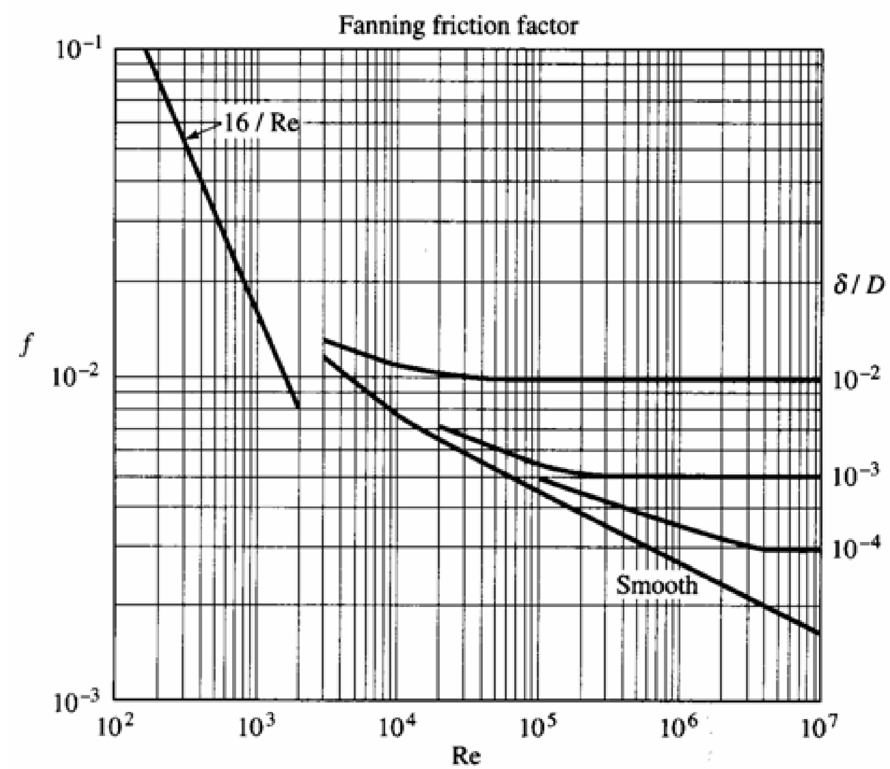
Mass balance $\quad w_1 = w_2 \qquad U_1 = U_2$

$$\Delta\frac{U^2}{2} + g\Delta h + \int_{p_1}^{p_2}\frac{dp}{\rho} = \hat{W}_s - \hat{E}_v$$

$$p_2 - p_1 = \frac{-\hat{E}_v}{UA}$$

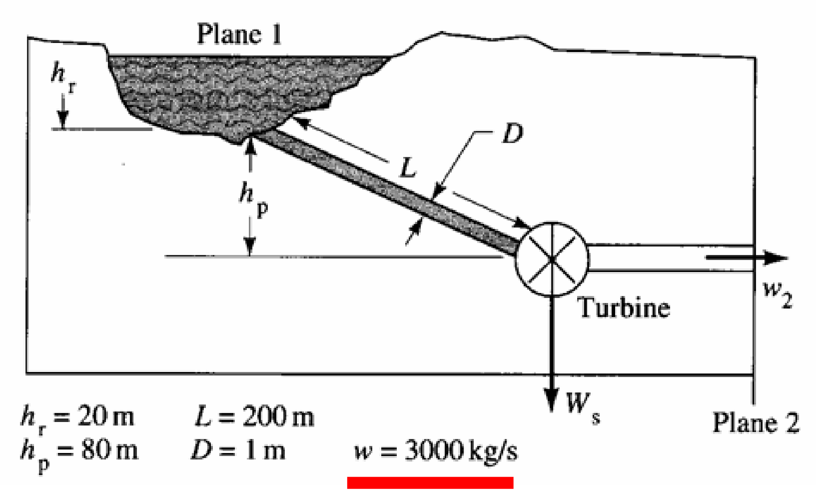
$$f \equiv \frac{D(p_1 - p_2)}{2L\rho U^2}$$

$$e_v \equiv \frac{\hat{E}_v}{U^2/2} = \frac{4L}{D}f$$



Pressure drop is associated with an irreversible loss of energy due to friction (viscous effects)

An elevated reservoir as a power source



Bernoulli equation

$$\frac{U_2^2 - U_1^2}{2} + g(h_2 - h_1) = \hat{W}_s - \hat{E}_v$$

$$-\hat{W}_s = \frac{(U_1^2 - U_2^2)}{2} + g h_1 - \hat{E}_v$$

Assume no frictional loss (maximum power) and negligible kinetic energy to the downstream fluid

$$-\hat{W}_s = g h_1 = 9.8(20 + 80) = 980 \text{ m}^2/\text{s}^2$$

$$-W_s = -w\hat{W}_s = 3000 \text{ kg/s} \times 980 \text{ m}^2/\text{s}^2 = 2940 \text{ kW} = 3940 \text{ hp}$$

In the absence of the turbine but given significant frictional loss

$$\frac{U_2^2 - U_1^2}{2} + g(h_2 - h_1) = \hat{W}_s - \hat{E}_v \qquad e_v \equiv \frac{\hat{E}_v}{U^2/2} = \frac{4L}{D} f$$

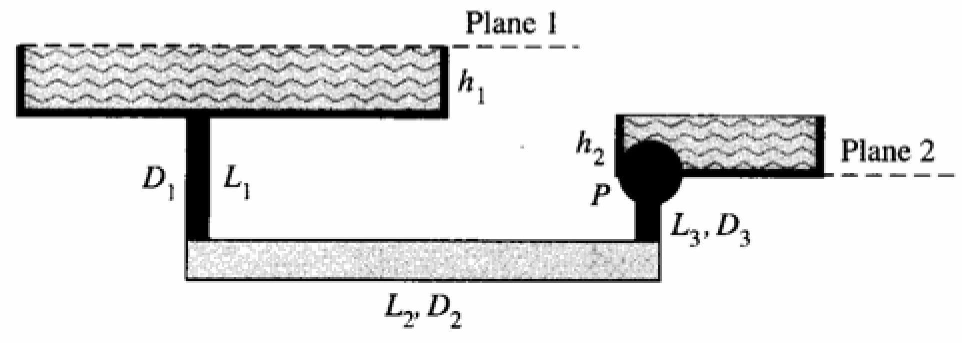
$$\frac{U_2^2}{2} - gh_1 + \frac{2LU_2^2}{D} f = 0 \qquad f = 0.079\,\mathrm{Re}^{-0.25} = 0.079\left(\frac{4w}{\pi D \mu}\right)^{-0.25} \qquad w = \rho U A$$

$$0.24 \frac{L\mu^{-0.25}}{\rho^2 D^{4.25}} w^{1.75} = gh_1 - \frac{0.81}{\rho^2 D^4} w^2 \quad \Rightarrow \quad w = 2.6 \times 10^4 \,\mathrm{kg/s}$$

trial and error

Flow rate is only 10% when there is a turbine. The difference arises from energy loss associated with turbine, and with valves that control overy the flow

Pump requirements for a water supply system



$h_1=10m$, $h_2=3m$,
$L_1=50m$, $L_2=300m$,
$L_3=1m$, $D_1=0.2m$,
$D_2=0.5m$, $D_3=0.03m$,
$w_2=60kg/s$

Is the pump necessary to supply water to the lower reservoir at a flow rate of 60*kg/s*?

Bernoulli equation

$$g(h_1 + L_1 - L_3) = \frac{U_3^2}{2} + gh_2 + \sum_i^n \hat{E}_{v,i}$$

At plane 1 At plane 2

$$\sum_i^3 \hat{E}_{v,i} = \sum_i^3 \frac{2L_i U_i^2}{D_i} f_i$$

Assume friction loss from straight pipes, not from bends and fittings

$$U_3^2 = \frac{2g(h_1 + L_1 - L_3 - h_2)}{1+\Phi}$$

$$D_i^2 U_i = \text{constant} = D_3^2 U_3$$

$$\Phi \equiv \frac{4L_3}{D_3} f_3 + \frac{4L_2}{D_2}\left(\frac{D_3}{D_2}\right)^4 f_2 + \frac{4L_1}{D_1}\left(\frac{D_3}{D_1}\right)^4 f_1$$

Implicit in U_3 -> need iterative method

First, assume the friction losses are minor

$$\tilde{U}_3 = [2g(h_1 + L_1 - L_3 - h_2)]^{1/2} = [2 \times 9.8 \times 56]^{1/2} = 33 \text{ m/s}$$

$$\tilde{w}_3 = \rho \frac{\pi D_3^2}{4} \tilde{U}_3 = 1000 \frac{\pi (0.03)^2}{4} 33 = 23.4 \text{ kg/s}$$

$$1 + \Phi = 1 + 132 f_3 + \underline{0.03 f_2 + 0.51 f_1}$$

$$\text{Re}_3 = 10^6 \quad \text{Re}_2 = 6 \times 10^4 \quad \text{Re}_1 = 1.5 \times 10^5$$

$$f = 0.079\,\text{Re}^{-0.25} \quad \text{for Re} > 10^4$$

$$f_3 = 2.5 \times 10^{-3} \quad f_2 = 5 \times 10^{-3} \quad f_1 = 4 \times 10^{-3}$$

$$U_3 = \left(\frac{33^2}{1+0.33} \right)^{1/2} = 28.6 \text{ m/s} \quad \text{and} \quad w_3 = 20 \text{ kg/s}$$

Only 1/3 of the desired mass flow
-> need a pump

$$\frac{U_3^2}{2} = g(h_1 + L_1 - L_3) - gh_2 - \sum_i^n \hat{E}_{v,i} + \hat{W}_s$$

$$+\hat{W}_s = -g(h_1 + L_1 - L_3) + gh_2 + \frac{U_3^2(1+\Phi)}{2}$$

$$+\hat{W}_s = 4100 \text{ m}^2/\text{s}$$

$$\hat{W}_s = w\hat{W}_s = 60 \text{ kg/s} \times 4100 \text{ m}^2/\text{s} = 243 \text{ kW} = 326 \text{ hp}$$

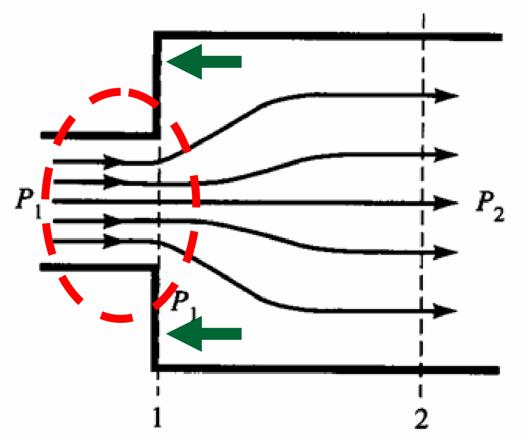
Macroscopic balance equations

$$\frac{dM}{dt} = [\rho U A]_{\text{in}} - [\rho U A]_{\text{out}} = \rho_1 U_1 A_1 - \rho_2 U_2 A_2$$

$$\frac{\partial}{\partial t}\int \rho \mathbf{u}\, dV = \rho_1 \langle U_1^2 \rangle \mathbf{A}_1 - \rho_2 \langle U_2^2 \rangle \mathbf{A}_2 + P_1 \mathbf{A}_1 - P_2 \mathbf{A}_2 - \mathbf{F} + M\mathbf{g}$$

$$\Delta \frac{U^2}{2} + g\,\Delta h + \int_{p_1}^{p_2} \frac{dp}{\rho} = \hat{W}_{\text{s}} - \hat{E}_{\text{v}}$$

Energy loss through bends and fittings



$$\Delta\frac{U^2}{2} + g\Delta h + \int_{p_1}^{p_2}\frac{dp}{\rho} = \hat{W}_s - \hat{E}_v$$

$$\hat{E}_v = \frac{U_1^2 - U_2^2}{2} + \frac{P_1 - P_2}{\rho}$$

Pressure unknown
-> need momentum balance

$$w_1U_1 - w_2U_2 + P_1A_1 - P_2A_2 = \underline{F}$$

$$w_1U_1 - w_2U_2 + P_1A_1 - P_2A_2 = -(A_2 - A_1)P_1$$

Assume negligible friction loss along the surface

$$(P_2 - P_1)A_2 = w(U_1 - U_2) = \rho U_2 A_2 (U_1 - U_2)$$

$$\hat{E}_v = \frac{U_2^2}{2}(\beta_{21} - 1)^2 = \frac{(U_1 - U_2)^2}{2} \qquad e_v \equiv \frac{\hat{E}_v}{U^2/2} \qquad U_1 = U_2\frac{A_2}{A_1} = \beta_{21}U_2$$

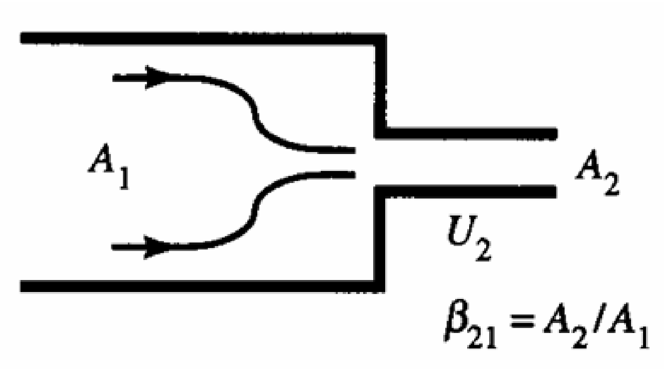
$$e_v = (\beta_{21} - 1)^2$$

Friction loss factor
Defined based on downstream velocity U_2

$$\hat{E}_{\mathrm{v}} = \frac{U_1^2 - U_2^2}{2} + \frac{P_1 - P_2}{\rho}$$

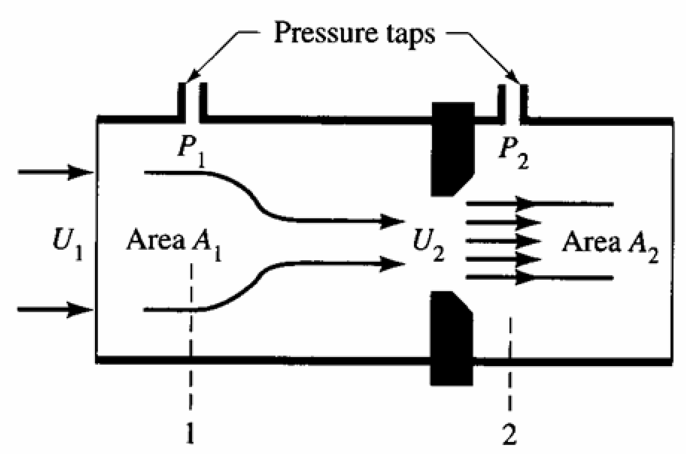
$$\rho g h_{\mathrm{L}} = \frac{\rho(U_1^2 - U_2^2)}{2} + (P_1 - P_2)$$

Hydrostatic head <- friction loss



$$\hat{E}_{\mathrm{v}} = \frac{U_2^2 e_{\mathrm{v}}}{2} \text{ with } e_{\mathrm{v}} = 0.45(1 - \beta_{21}).$$

Flow measuring device: orifice meter



$$\hat{E}_v = \frac{U_1^2 - U_2^2}{2} + \frac{P_1 - P_2}{\rho}$$

Assume no friction loss

$$U_1 = \beta\left[\frac{2(P_1 - P_2)}{\rho(1-\beta^2)}\right]^{1/2} \qquad \beta \equiv \frac{A_2}{A_1} < 1$$

$$Q = A_2 U_2 = A_1 U_1 = A_2\left[\frac{2(P_1 - P_2)}{\rho(1-\beta^2)}\right]^{1/2}$$

Larger than the actual flow rate

$$Q = C_d A_2\left[\frac{2(P_1 - P_2)}{\rho(1-\beta^2)}\right]^{1/2}$$