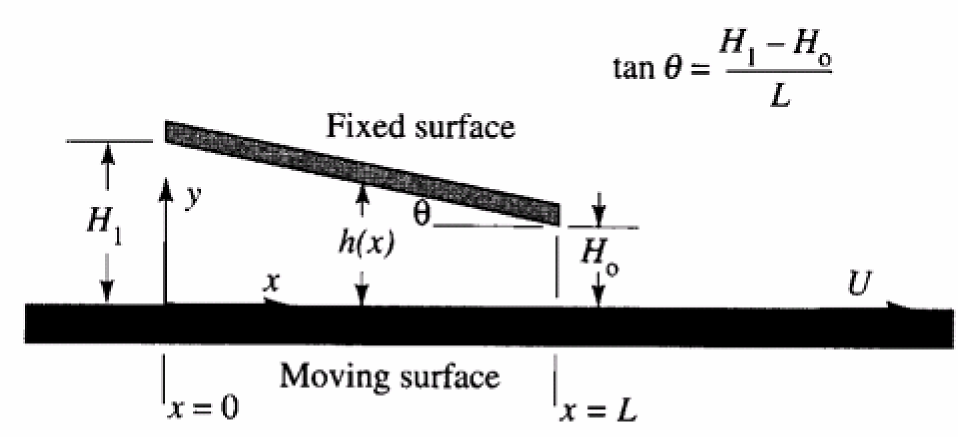
Chapter 6

Nearly parallel flows

Slider bearing

Steady state, isothermal, incompressible, Newtonian fluid, small angle



$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right)$$

$$\frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \frac{\partial \tilde{u}_y}{\partial \tilde{y}} = 0$$

$$\cancel{\frac{\rho U H_1}{\mu}}\left(\tilde{u}_x \frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \tilde{u}_y \frac{\partial \tilde{u}_x}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \left(\frac{\partial^2 \tilde{u}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}_x}{\partial \tilde{y}^2}\right)$$

$$\cancel{\frac{\rho U H_1}{\mu}}\left(\tilde{u}_x \frac{\partial \tilde{u}_y}{\partial \tilde{x}} + \tilde{u}_y \frac{\partial \tilde{u}_y}{\partial \tilde{y}}\right) = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \left(\frac{\partial^2 \tilde{u}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}_y}{\partial \tilde{y}^2}\right)$$

$$\tilde{x} = \frac{x}{H_1} \qquad \tilde{y} = \frac{y}{H_1}$$

$$\tilde{u}_x = \frac{u_x}{U} \qquad \tilde{u}_y = \frac{u_y}{U} \qquad \tilde{p} = \frac{p H_1}{\mu U} \quad \left(\rho U^2\right)$$

$$\frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \frac{\partial \tilde{u}_y}{\partial \tilde{y}} = 0 \qquad 0 = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \left(\frac{\partial^2 \tilde{u}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}_x}{\partial \tilde{y}^2} \right) \qquad 0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \left(\frac{\partial^2 \tilde{u}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}_y}{\partial \tilde{y}^2} \right)$$

<u>Parallel flow</u> $u_x = u(y)$

$$\frac{\partial \tilde{u}_x}{\partial \tilde{x}} = 0 \qquad 0 = -\frac{d\tilde{p}}{d\tilde{x}} + \frac{d^2 \tilde{u}_x}{d\tilde{y}^2} \qquad 0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

$$\Rightarrow \quad \boxed{\tilde{u}_x = 1 - \tilde{y}}$$

<u>Nearly parallel flow</u>

$$0 = -\frac{d\tilde{p}}{d\tilde{x}} + \frac{\partial^2 \tilde{u}_x}{\partial \tilde{y}^2} \qquad 0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

Velocity gradient in the *x* direction is much smaller than that in the *y* direction

$$\tilde{u}_x = 0 \quad \text{on} \quad \tilde{y} = \eta(\tilde{x})$$

$$\tilde{u}_x = 1 \quad \text{on} \quad \tilde{y} = 0$$

$$\Rightarrow \quad \boxed{\tilde{u}_x = \left(1 - \frac{\tilde{y}}{\eta}\right) - \frac{\eta^2}{2}\frac{d\tilde{p}}{d\tilde{x}}\left[\frac{\tilde{y}}{\eta} - \left(\frac{\tilde{y}}{\eta}\right)^2\right]}$$

$$\eta = \frac{h(x)}{H_1} = 1 - \tilde{x}\tan\theta$$

$$\frac{Q}{W} = \int_0^{h(x)} u_x(x,y)\,dy = UH_1 \int_0^{\eta} \tilde{u}_x(\tilde{x},\tilde{y})\,d\tilde{y}$$

$$\lambda \equiv \frac{Q}{WUH_1} = \int_0^{\eta} \tilde{u}_x(\tilde{x},\tilde{y})\,d\tilde{y} = \frac{\eta}{2} - \frac{\eta^3}{12}\frac{d\tilde{p}}{d\tilde{x}}$$

Unknown
dimensionless const.

$$\frac{d\tilde{p}}{d\tilde{x}} = 12\left[\frac{1}{2\eta^2} - \frac{\lambda}{\eta^3}\right] \qquad \tilde{p}(\tilde{x}) = \tilde{p}(0) + 6\int_0^{\tilde{x}} \frac{d\tilde{x}}{\eta^2} - 12\lambda \int_0^{\tilde{x}} \frac{d\tilde{x}}{\eta^3}$$

$$\tilde{p}(\tilde{x}) = 0 \quad \text{at} \quad \tilde{x} = 0 \quad \text{and} \quad \tilde{x} = \frac{L}{H_1} \equiv \Lambda \qquad \Lambda = \frac{1-\kappa}{\tan\theta}, \quad \kappa = \frac{H_o}{H_1}$$

Pressure is atmospheric at both ends of the bearing

$$\lambda = \frac{1}{2}\left[\frac{\int_0^{\Lambda} \frac{d\tilde{x}}{\eta^2}}{\int_0^{\Lambda} \frac{d\tilde{x}}{\eta^3}}\right] \qquad \lambda = \frac{\kappa}{1+\kappa} \qquad \tilde{p} = \frac{6\tilde{x}(\eta-\kappa)}{\eta^2(1+\kappa)}$$

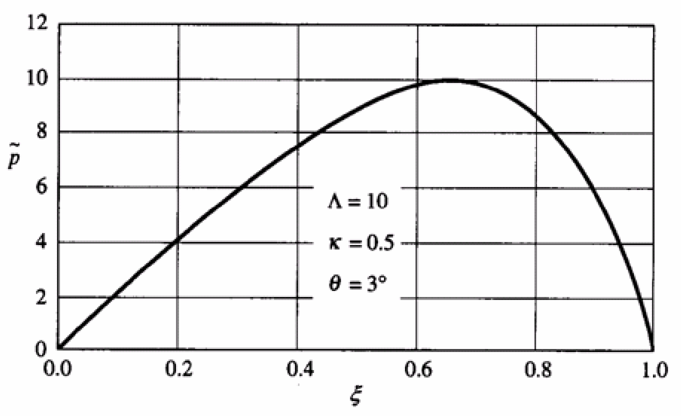
$$\eta = \frac{h(x)}{H_1} = 1 - \tilde{x}\tan\theta$$

$$\frac{d\tilde{p}}{d\tilde{x}} = 12\left[\frac{1}{2\eta^2} - \frac{\lambda}{\eta^3}\right]$$

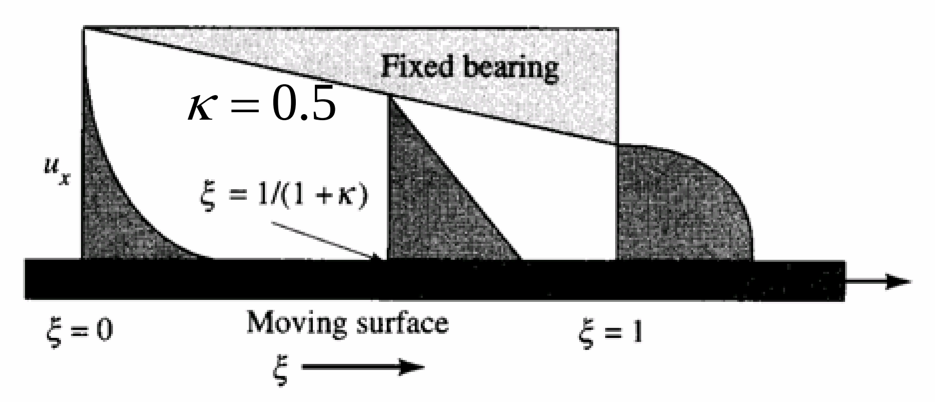
Maximum pressure

$$\eta = 2\lambda \qquad \xi \equiv \frac{x}{L} = \frac{1}{1+\kappa}$$

$$\tilde{p}^{\max} = \frac{3\Lambda}{2}\frac{1-\kappa}{\kappa(1+\kappa)}$$



Max. pressure increases with increasing viscosity, speed, bearing length; why?



At max.pressure -> drag flow
before max. -> against the flow
after max. -> aids the flow

Check velocity profile at entrance with $\kappa < 0.5$

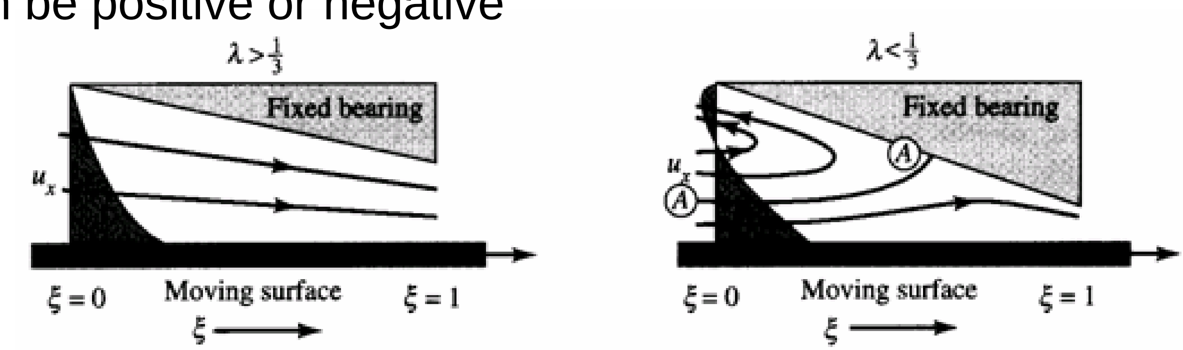
Effect of geometry on the character of the flow

$$\eta = \frac{h(x)}{H_1} = 1 - \tilde{x}\tan\theta$$

At the entrance $\left.\frac{\partial \tilde{u}_x}{\partial \tilde{y}}\right|_{\eta=1} = 2 - 6\lambda$

$$\lambda \equiv \frac{Q}{WUH_1} = \int_0^{\eta} \tilde{u}_x(\tilde{x},\tilde{y})\, d\tilde{y} = \frac{\eta}{2} - \frac{\eta^3}{12}\frac{d\tilde{p}}{d\tilde{x}}$$

Velocity gradient can be positive or negative



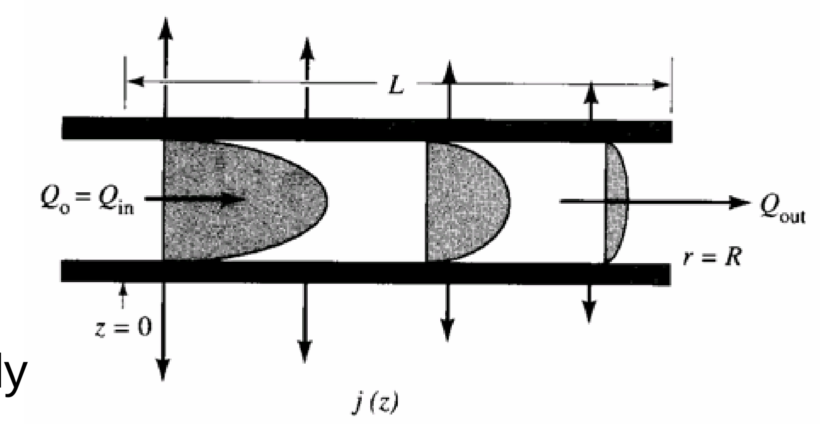
When the gap (H_o) becomes too small;

1. surface roughness causes solid-solid contact -> wear and damage
2. flow may change from laminar to turbulent
3. too high pressure may change the viscosity or bearing surface
4. frictional heating -> reduces viscosity, thermal degradation

➡ Lubrication theory fails

Flow through a leaky tube

Goal: to develop a model with which to determine the fractional leakage associated with the flow, and the dependence of the leakage on design and operating parameters



Steady state, axisymmetric, low Re, uniformly permeable wall

$$u = [u_r(r,z), 0, u_z(r,z)], \quad p = p(r,z)$$

$$j(z) = K[p(z) - p_a] \qquad j(z) = u_r(R, z) = u_R(z)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_r}{\partial z} \qquad 0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial^2 u_r}{\partial z^2}\right] \qquad 0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right]$$

If leakage flow is small, could the axial velocity have a functional dependence on radial position nearly the same as in the case of no leakage, but with a minor correction that accounts for leakage?

$$u_z = 2U(z)\left[1 - \left(\frac{r}{R}\right)^2\right] = 2U_o\left[1 - \left(\frac{r}{R}\right)^2\right] f(z)$$

$$u_r = -\frac{1}{r}\int_0^r r\frac{\partial u_z}{\partial z}dr \qquad u_r = -RU_o f'(z)\left[\left(\frac{r}{R}\right)^1 - \frac{1}{2}\left(\frac{r}{R}\right)^3\right] \qquad u_r(R, z) \equiv u_R(z) = -\frac{RU_o}{2}f'$$

$$\frac{\partial p}{\partial z} = \mu\left[-\frac{8U_o}{R^2}f + 2U_o\left(1-\left(\frac{r}{R}\right)^2\right)f''\right] \qquad \frac{\partial p}{\partial z} = \frac{1}{K}\frac{du_R}{dz} = -\frac{RU_o}{2K}f''$$

$$\boxed{f'' = \frac{16K\mu}{R^3}f - \frac{4K\mu}{R}\left[1-\left(\frac{r}{R}\right)^2\right]f''}$$

f is a function of both *z* and *r*, which violates original assumption *f=f(z)*

OK if the leakage is small (2nd term is small) $\frac{4K\mu}{R} << 1$

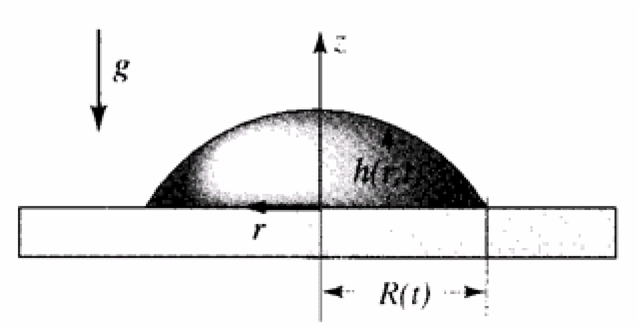
$$f'' - \frac{16K\mu}{R^3}f = 0$$

$$f = 1 \quad \text{at} \quad z = 0$$

$$f \text{ remains finite} \qquad \text{for} \quad z \to \infty$$

$$\Rightarrow \quad f = \exp\left[-4\beta\frac{z}{R}\right] \qquad \beta = \left(\frac{K\mu}{R}\right)^{1/2}$$

Spreading of a very viscous drop



Drop of Volume V

Neglect surface tension and inerial effects unsteady, but creeping flow

Goal: to develop a model for the radius of the drop as a function of time *R(t)*

Assume quasi-steady, gravity/viscous dominated, nearly parallel flow

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right) + \rho g_r \quad \text{(4.3.24d)}$$

Nearly parallel

$$p = p_o + \rho g(h - z) \qquad g\frac{\partial h}{\partial r} = \nu\frac{\partial^2 u_r}{\partial z^2}$$

$$u_r = 0 \quad \text{on} \quad z = 0$$

$$\tau_{rz} = -\mu\frac{\partial u_r}{\partial z}\bigg|_{z=h} = 0 \quad \Rightarrow \quad u_r = -\frac{g}{2\nu}\frac{\partial h}{\partial r}z(2h - z)$$

$$\frac{\partial u_z}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r} r u_r = 0 \qquad [u_z(h)-\cancel{u_z}(0)]+\frac{1}{r}\frac{\partial}{\partial r} r\int_0^h u_r\,dz = 0 \qquad \frac{\partial h}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\int_0^h u_r\,dz\right]=0$$

impermeable

$$\boxed{\frac{\partial h}{\partial t}-\frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3\frac{\partial h}{\partial r}\right)=0}$$

Nonlinear PDE

$$h = 0\; at\; r = R, \quad \partial h/\partial r = 0\; at\; r = 0$$

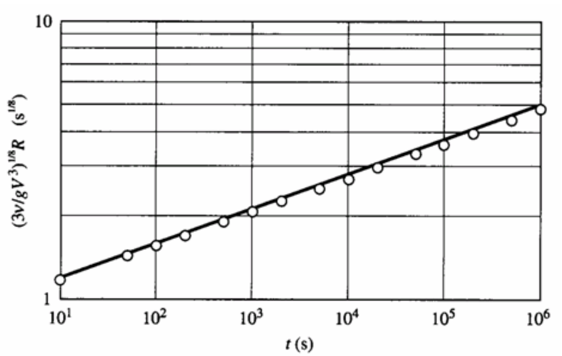
$$h(t) = 0.531\underline{\left(\frac{3\nu V}{gt}\right)^{1/4}}\left[1-\left(\frac{r}{R}\right)^2\right]^{1/3} \qquad 2\pi\int_0^{R(t)} r h\,dr = V \qquad \boxed{\left(\frac{3\nu}{gV^3}\right)^{1/8} R(t) = 0.894\,t^{1/8}}$$

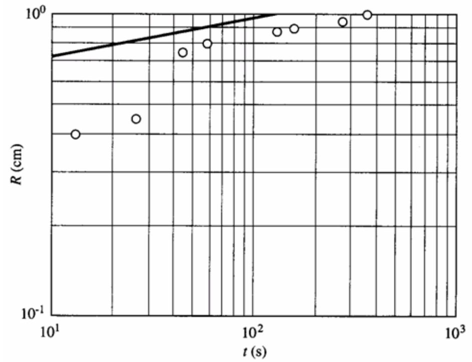
Not for small *t*

<u>Criterion</u>

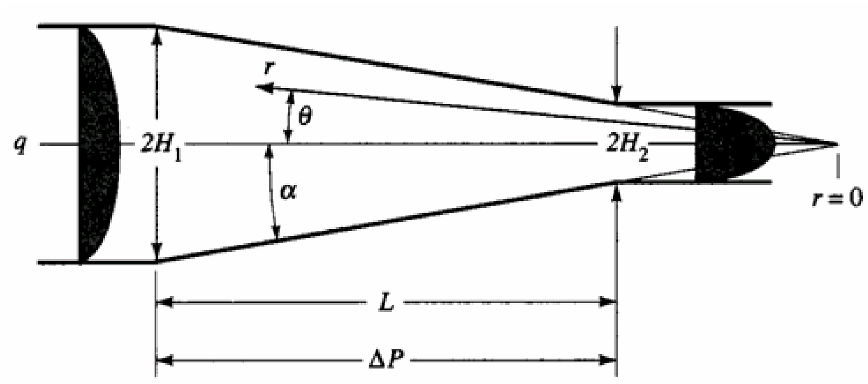
$$\frac{h(t, r=0)}{R(t)} < 0.1$$

$$\frac{V^{1/3} g t}{\nu} \geq 350$$





Flow through a converging planar region



Parallel flow

$$u_z = \frac{-CH^2}{2\mu}\left[1 - \frac{y^2}{H^2}\right]$$

$$-C = \frac{\Delta P}{L} = \frac{3\mu Q}{2WH^3}$$

Neglect entrance/exit effect; $L >> H_1$

Strictly radial flow

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \qquad r u_r \neq g(r) \qquad \boxed{r u_r = f(\theta)}$$

$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r \frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \cancel{\frac{u_\theta^2}{r}} + \cancel{u_z \frac{\partial u_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 u_r}{\partial z^2}}\right) + \cancel{\rho g_r} \quad \text{(4.3.24d)}$$

$$\rho\left(\cancel{\frac{\partial u_\theta}{\partial t}} + \cancel{u_r \frac{\partial u_\theta}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \cancel{\frac{u_r u_\theta}{r}} + \cancel{u_z \frac{\partial u_\theta}{\partial z}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\cancel{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]} + \cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \cancel{\frac{\partial^2 u_\theta}{\partial z^2}}\right) + \cancel{\rho g_\theta} \quad \text{(4.3.24e)}$$

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^3}\frac{d^2 f}{d\theta^2} + \frac{\rho f^2}{r^3} \qquad \frac{\partial p}{\partial \theta} = \frac{2\mu}{r^2}\frac{df}{d\theta}$$

Cross differentiation and substraction

$$\frac{d^3 f}{d\theta^3} + \left(4 + \frac{2\rho}{\mu} f\right)\frac{df}{d\theta} = 0$$

no slip on the planes $\theta = \pm\alpha$: $f(\alpha) = f(-\alpha) = 0$

$$q = \int_{-\alpha}^{\alpha} u_r r \, d\theta = \int_{-\alpha}^{\alpha} f(\theta) \, d\theta$$

Negative in the converging flow

$$\int_{-\alpha}^{\alpha} f(\theta) \, d\theta = 2\alpha \bar{f} = q$$

Mean value theorem

Characteristic value of f is of the order of q/α

Nondimensionalize

$$\phi = \frac{\theta}{\alpha} \qquad F = \frac{\alpha f}{q}$$

$$\frac{d^3 F}{d\phi^3} + \left(4\alpha^2 + \mathrm{Re}\, F\right)\frac{dF}{d\phi} = 0$$

$$\mathrm{Re} = \frac{2\rho q \alpha}{\mu}$$

no slip on the planes $\phi = \pm 1$: $F(1) = F(-1) = 0$

$$\int_{-1}^{1} F(\phi) \, d\phi = 1$$

For small Reynolds number $\quad \mathrm{Re}\, F\dfrac{dF}{d\phi} << \dfrac{d^3F}{d\phi^3}$

$$\frac{d^3F}{d\phi^3} + 4\alpha^2\frac{dF}{d\phi} = 0 \qquad Y = \frac{dF}{d\phi} \qquad \frac{d^2Y}{d\phi^2} + 4\alpha^2 Y = 0$$

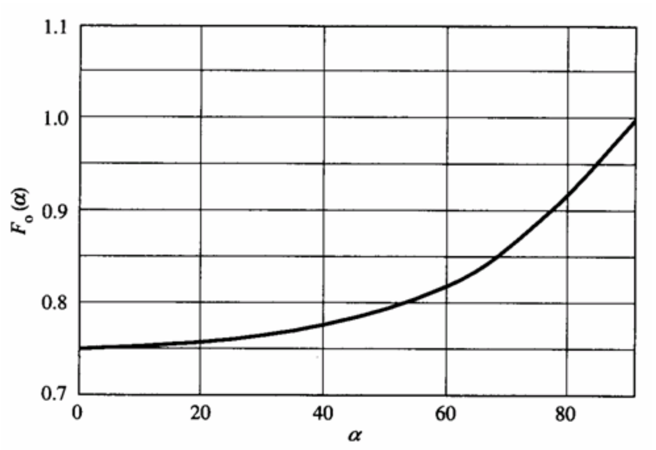
$$Y = \frac{dF}{d\phi} = A'\sin 2\alpha\phi + B'\cos 2\alpha\phi \qquad F = A\sin 2\alpha\phi + B\cos 2\alpha\phi + C$$

$$\boxed{F(\phi) = \frac{\alpha(\cos 2\alpha\phi - \cos 2\alpha)}{\sin 2\alpha - 2\alpha\cos 2\alpha}}$$

For parallel flow $\quad \alpha = 0 \qquad F(\phi) = \frac{3}{4}(1-\phi^2)$

$$\frac{d^3F}{d\phi^3} = 0 \quad F(\phi) = A + B\phi + C\phi^2$$

Nearly parallel flow



Maximum (centerline) velocity

Flow is almost parallel up to an angle of 20^o