Chapter 8

Stream function

2-dimensional flow

$$u_x = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad u_y = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \qquad \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

Continuity is satisfied by itself

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right)$$

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right)$$

Cross differentiation

$$\frac{\partial \psi}{\partial x}\frac{\partial(\nabla^2 \psi)}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial(\nabla^2 \psi)}{\partial x} = \nu \nabla^4 \psi$$

$$\nabla^2 \psi \equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\nabla^4 \psi \equiv \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} \frac{\partial^4 \psi}{\partial y^4}$$

3 equations into 1, but formidable

Geometric interpretation

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -u_x \, dx + u_x \, dy$$

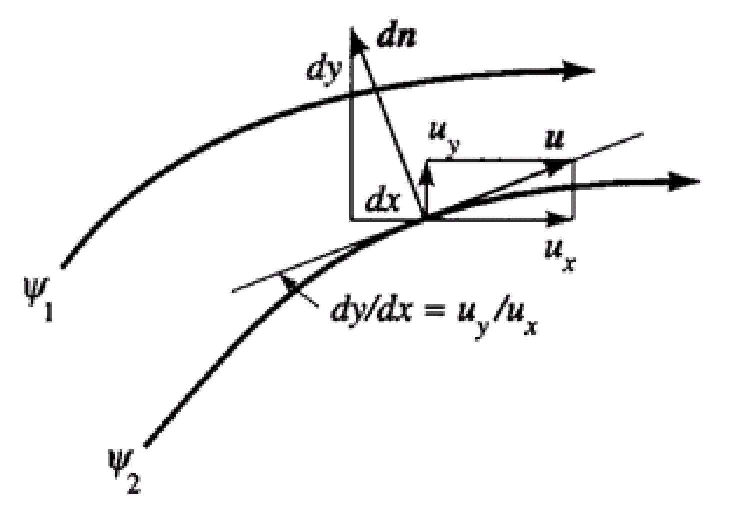
Along a line of constant ψ for which $d\psi = 0$

$$\left[\frac{dy}{dx}\right]_{\psi_1} = \frac{u_y}{u_x}$$

Streamline: a line whose tangent at any point along the streamline is collinear with the velocity vector at that point (the path of a ‘tracer particle’)

$$\dot{m} = \int_{\psi_1}^{\psi_2} \rho \, d\psi = \rho(\psi_2 - \psi_1)$$

Mass flow between a pair of streamlines is proportional to its difference
If the values differ by a constant increment, the velocity is higher when the stramlines are closer



Poiseuille flow in a tube

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \quad \text{and} \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$0 = -\frac{\partial p}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + \frac{\partial^2 u_r}{\partial z^2}\right\} \qquad 0 = -\frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right\}$$

$$0 = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right)$$

Fully developed flow -> ψ is a function only of the radial coordinate *r*

$$0 = \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr}\right) \qquad \psi = r^n \qquad \psi = a + br^2 + cr^4$$

$$u_z = 0 \quad \text{on} \quad r = R; \quad \frac{du_z}{dr} = 0 \quad \text{on} \quad r = 0 \quad \Rightarrow \quad c = -\frac{1}{2}\frac{b}{R^2}$$

$$-\frac{1}{r}\frac{d\psi}{dr} = 0 \quad \text{on} \quad r = R; \quad \frac{d}{dr}\left(\frac{1}{r}\frac{d\psi}{dr}\right) = 0 \quad on \quad r = 0$$

Automatically satisfied

$$q_{12} = \int_{r_1}^{r_2} 2\pi r u_z(r)\, dr = \int_{r_1}^{r_2} 2\pi \frac{d\psi}{dr} dr = 2\pi[\psi(r_2) - \psi(r_1)]$$

Difference between ψ at two radii is proportional to the flowrate through the annulus bounded by two radii

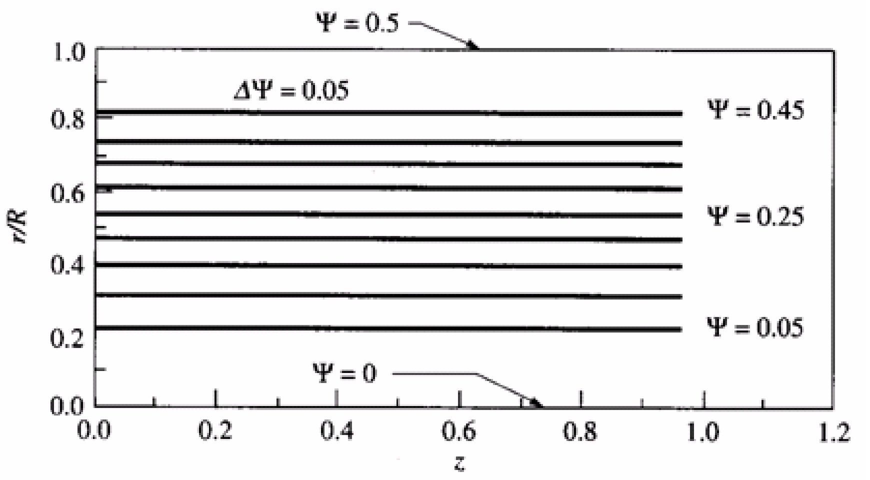
$$\psi(R) - \psi(0) = \frac{Q}{2\pi} = \frac{UR^2}{2} \quad \Rightarrow \quad b = U \qquad \psi = \underline{a} + UR^2\left[\left(\frac{r}{R}\right)^2 - \frac{1}{2}\left(\frac{r}{R}\right)^4\right]$$

Arbitrary and free to choose (*a*=*0*)

$$\psi = UR^2\left[\left(\frac{r}{R}\right)^2 - \frac{1}{2}\left(\frac{r}{R}\right)^4\right] = \frac{Q}{\pi}\left[\left(\frac{r}{R}\right)^2 - \frac{1}{2}\left(\frac{r}{R}\right)^4\right]$$

Nondimensionalize

$$\frac{\pi\psi}{Q} \equiv \Psi = \left(\frac{r}{R}\right)^2 - \frac{1}{2}\left(\frac{r}{R}\right)^4$$



Flow around a sphere/ flow in a CVD reactor
Flow in an occluded blood vessel/ radial flow between parallel disks