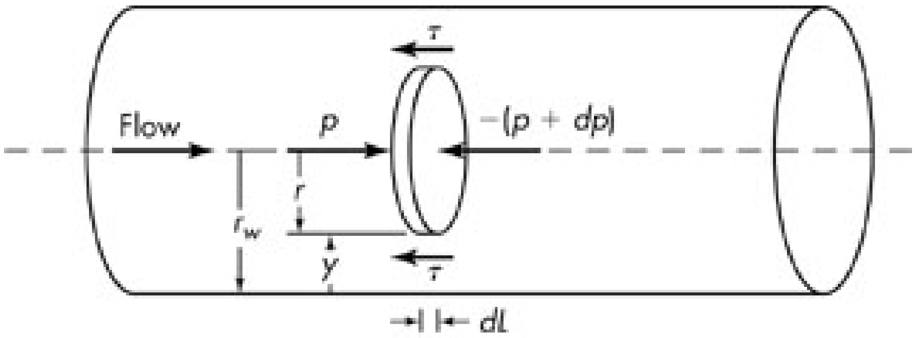
Chapter 5. Incompressible Flow in Pipes and Channels

Shear Stress and Skin Friction in Pipes (전단응력 및 표면마찰)

* Shear-stress distribution



For fully developed flow, $\overline{V}_b = \overline{V}_a$ & $\beta_b = \beta_a$

$$\therefore \sum F = 0 \quad \leftarrow \quad \text{from Eq. (4.51):} \ \sum F = \dot{m}\left(\beta_b \overline{V}_b - \beta_a \overline{V}_a\right)$$

From Eq. (4.52), $\sum F = p_a S_a - p_b S_b + \underbrace{F_w}_{= -F_s} - \cancel{F_g} = 0$

$$\longrightarrow \quad \sum F = \pi r^2 p - \pi r^2 (p + dp) - (2\pi r dL)\tau = 0$$

$\pi r^2 dL$ 로 나누면

$$\frac{dp}{dL} + \frac{2\tau}{r} = 0 \qquad \text{--- Eq. (5.1)}$$

$\downarrow$

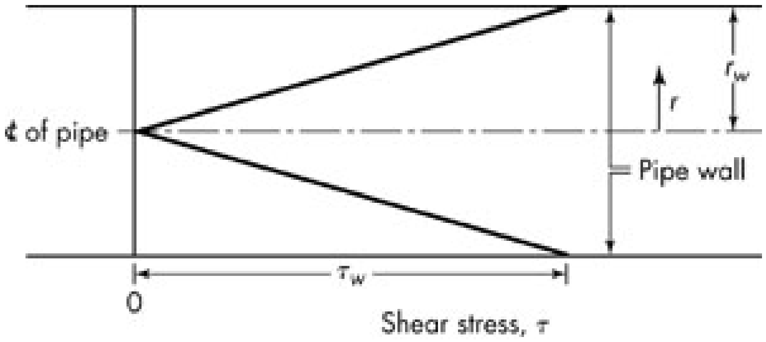
r 과는 무관 (관의 단면적 방향으로의 압력은 일정)

& $\tau = \tau_w$ at $r = r_w$

$$\therefore \ \frac{dp}{dL} + \frac{2\tau_w}{r_w} = 0 \qquad \text{--- Eq. (5.2)}$$

Eq. (5.2)에서 Eq. (5.1)을 빼면,

$$\frac{\tau_w}{r_w} = \frac{\tau}{r} \quad \text{or} \quad \boxed{\tau = \left(\frac{\tau_w}{r_w}\right) r}$$



* Relation between skin friction & wall shear

펌프에 의한 일이 없고 마찰을 고려할 경우의 Bernoulli 방정식은

$$\frac{p_a}{\rho}+gZ_a+\frac{\alpha_a\overline{V}_a^{\,2}}{2}=\frac{p_b}{\rho}+gZ_b+\frac{\alpha_b\overline{V}_b^{\,2}}{2}+h_f \qquad \text{--- Eq. (4.71)}$$

일반적으로 $p_a > p_b$ 이므로 $p_b = p_a - \Delta p$ 로 표시할 수 있고 fully developed flow인 수평관을 대상으로 하며 마찰은 유체와 관벽 사이의 skin friction h_{fs}만 존재하므로

$\overline{V}_b=\overline{V}_a,\ \alpha_b=\alpha_a,\ Z_b=Z_a$, & $\Delta p=\Delta p_s$ (압력강하는 표면마찰에 의한 것이므로)

이 경우 Bernoulli 식은

$$\frac{p_a}{\rho}=\frac{p_a-\Delta p_s}{\rho}+h_{fs} \quad \text{즉,} \quad \frac{\Delta p_s}{\rho}=h_{fs} \qquad \text{--- Eq. (5.4)}$$

From Eq. (5.2), $\dfrac{-\Delta p_s}{L}+\dfrac{2\tau_w}{r_w}=0$

Δp_s 를 소거하면

$$\therefore\ h_{fs}=\frac{2}{\rho}\frac{\tau_w}{r_w}L=\frac{4}{\rho}\frac{\tau_w}{D}L$$

*** Friction factor (마찰계수), f**

← 여기서 정의하는 마찰계수 f 는 *Fanning friction factor*

또다른 마찰계수로 *Blasius* or *Darcy friction factor*가 있는데 이는 $4f$ 에 해당

$$f \equiv \frac{\tau_w}{\rho \bar{V}^2/2} = \frac{2\tau_w}{\rho \bar{V}^2} \quad \text{--- Eq. (5.6)}$$

$$\frac{\text{wall shear stress}}{\text{density} \times \text{velocity head}} \quad \text{즉,} \quad \frac{(\text{단위면적당 전단력})}{(\text{단위부피당 운동에너지})}$$

skin friction h_{fs} 와 friction factor f 와의 관계:

$$h_{fs} = \frac{2}{\rho}\frac{\tau_w}{r_w} L = \frac{\Delta p_s}{\rho} = 4f \frac{L}{D}\frac{\bar{V}^2}{2} \quad \text{--- Eq. (5.7)}$$

$$\boxed{\therefore \; f = \frac{\Delta p_s D}{2L\rho \bar{V}^2}} \quad \text{or} \quad \frac{\Delta p_s}{L} = \frac{2 f \rho \bar{V}^2}{D} \quad \text{--- Eqs. (5.8)-(5.9)}$$

* Flow in noncircular channels

In evaluating the diameter in noncircular channels, an equivalent diameter (등가지름) D_{eq} is used.

$$\boxed{D_{eq} = 4r_H}$$

→ r_H : hydraulic radius (수력학적 반지름)

$$\boxed{r_H \equiv \frac{S}{L_p}}$$

S : cross-sectional area of channel
L_p : wetted perimeter

1) Circular tube:

$$r_H = \frac{\pi D^2/4}{\pi D} = \frac{D}{4}$$

→ $D_{eq}=D$

2) Annular pipes:

$$r_H = \frac{\pi D_o{}^2/4 - \pi D_i{}^2/4}{\pi D_i + \pi D_o} = \frac{D_o - D_i}{4}$$

→ $D_{eq}=D_o\text{-}D_i$

3) Square duct:

$$r_H = \frac{b^2}{4b} = \frac{b}{4}$$

→ $D_{eq}=b$

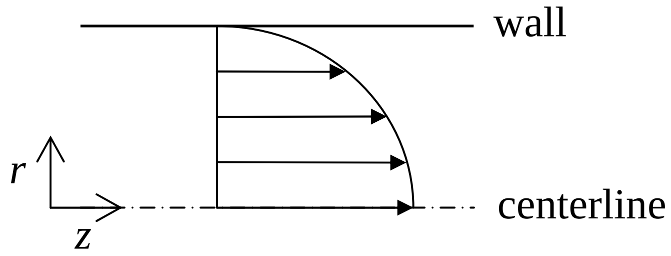
단면이 원형이 아닌 관의 경우 Reynolds number *Re* 또는 friction factor *f* 등의 계산시에 D 대신 D_{eq} 혹은 r 대신 $2r_H$ 를 대입하여 계산 가능함을 의미.

Laminar Flow in Pipes and Channels

* Laminar flow of Newtonian fluids

원형 단면을 갖는 흐름을 대상, 속도분포는 centerline에 대해 대칭

u depends only on r



$$\tau = -\mu \frac{du}{dr}$$

$$\frac{du}{dr} = -\frac{\tau}{\mu} = -\frac{\tau_w}{r_w \mu} r$$ ← Eq. (5.3) 적용

적분하면 (경계조건: $u = 0$ at $r = r_w$)

$$\int_0^u du = -\frac{\tau_w}{r_w \mu}\int_{r_w}^r r dr \longrightarrow u = \frac{\tau_w}{2 r_w \mu}\left(r_w{}^2 - r^2\right)$$ --- Eq. (5.15)

$$u_{\max} = \frac{\tau_w r_w}{2\mu} \quad (\text{at } r = 0)$$

$$\boxed{\therefore \frac{u}{u_{\max}} = 1 - \left(\frac{r}{r_w}\right)^2}$$ --- Eq. (5.17)

Average velocity

$$\bar{V} = \frac{1}{S}\int u dS$$ --- Eq. (4.11)

$dS = 2\pi r dr$

Eq. (5.15) 대입한 후 적분

$$\bar{V} = \frac{\tau_w}{r_w{}^3 \mu}\int_0^{r_w}\left(r_w{}^2 - r^2\right) r dr = \frac{\tau_w r_w}{4\mu}$$ --- Eq. (5.18)

이 식을 $u_{\max} = \dfrac{\tau_w r_w}{2\mu}$ 와 비교하면, $\boxed{\dfrac{\bar{V}}{u_{\max}} = 0.5}$ --- Eq. (5.19)

→ In Laminar flow,

Kinetic energy correction factor, $\alpha = 2.0$ $\longleftarrow$ Eq. (4.70)에 (5.15)와 (5.18)을 대입해 계산

Momentum correction factor, $\beta = \dfrac{4}{3}$ $\longleftarrow$ Eq. (4.50)에 (5.15)와 (5.18)을 대입해 계산

Hagen-Poiseuille equation

Eq. (5.7)과 Eq. (5.18)을 이용하여 τ_w 대신 보다 실제적인 Δp_s로 변환하면,

$$\bar{V} = \frac{\Delta p_s D^2}{32 L \mu} \quad \text{or} \quad \boxed{\Delta p_s = \frac{32 L \bar{V} \mu}{D^2}}$$ --- Eq. (5.20)

여기서 $q = \frac{\pi D^2}{4} \bar{V}$ 이므로 q 와 Δp_s 측정으로부터 점도 계산 가능:

$$\boxed{\mu = \frac{\pi \Delta p_s D^4}{128 L q}}$$

: Hagen-Poiseuille equation

또한 Eq. (5.7)에서 $\Delta p_s = 4\tau_w/(DL)$ 이므로,

$$\tau_w = \frac{8\bar{V}\mu}{D}$$ --- Eq. (5.21)

Eq. (5.21)을 Eq. (5.7)에 대입하면 ***f*** **와 Re 사이의 관계**가 유도됨:

$$\boxed{f = \frac{16\mu}{D\bar{V}\rho} = \frac{16}{\mathrm{Re}}}$$ --- Eq. (5.22)

* Laminar flow of non-Newtonian liquids

- Power law fluids

$$\tau = -K\frac{du}{dr}^n$$

반지름 r 에 따른 velocity profile:

$$u = \left(\frac{\tau_w}{r_w K}\right)^{1/n}\frac{r_w^{1+1/n} - r^{1+1/n}}{1+1/n}$$

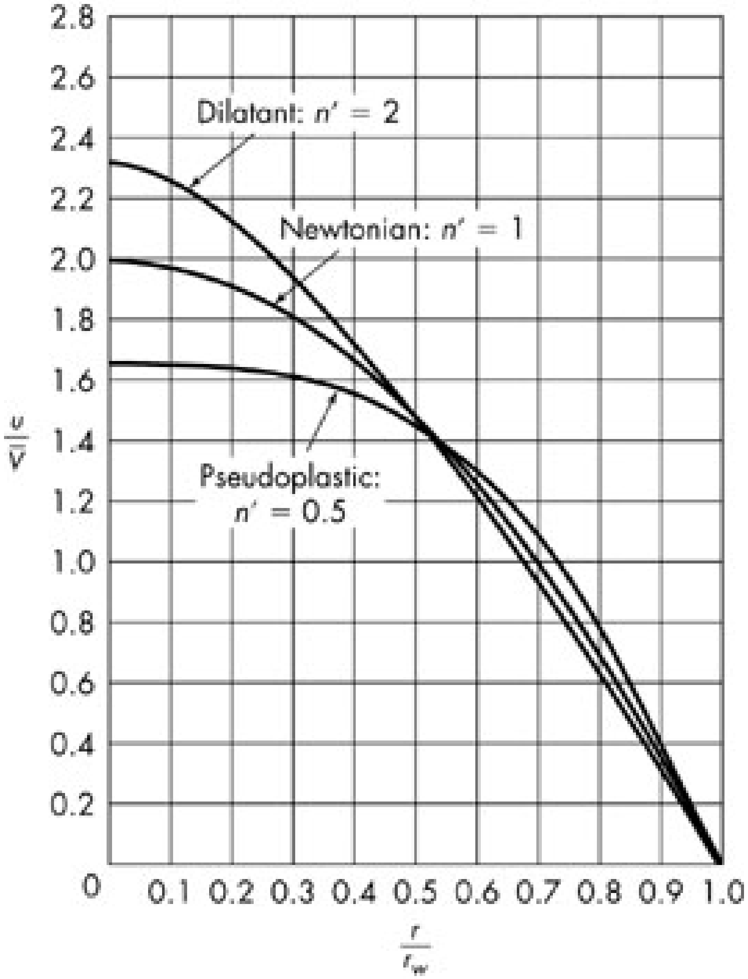


Fig. 5.4. Velocity profiles in the laminar flow of Newtonian and non-Newtonian liquids.

- Bingham model

$$\tau - \tau_o = -K\frac{du}{dr} \qquad \text{at } \tau > \tau_o$$

$$\frac{du}{dr} = 0 \qquad \text{at } \tau < \tau_o$$

반지름 r 에 따른 velocity profile:

$$u = \frac{1}{K}(r_w - r)\left[\frac{\tau_w}{2}\left(1+\frac{r}{r_w}\right) - \tau_0\right]$$

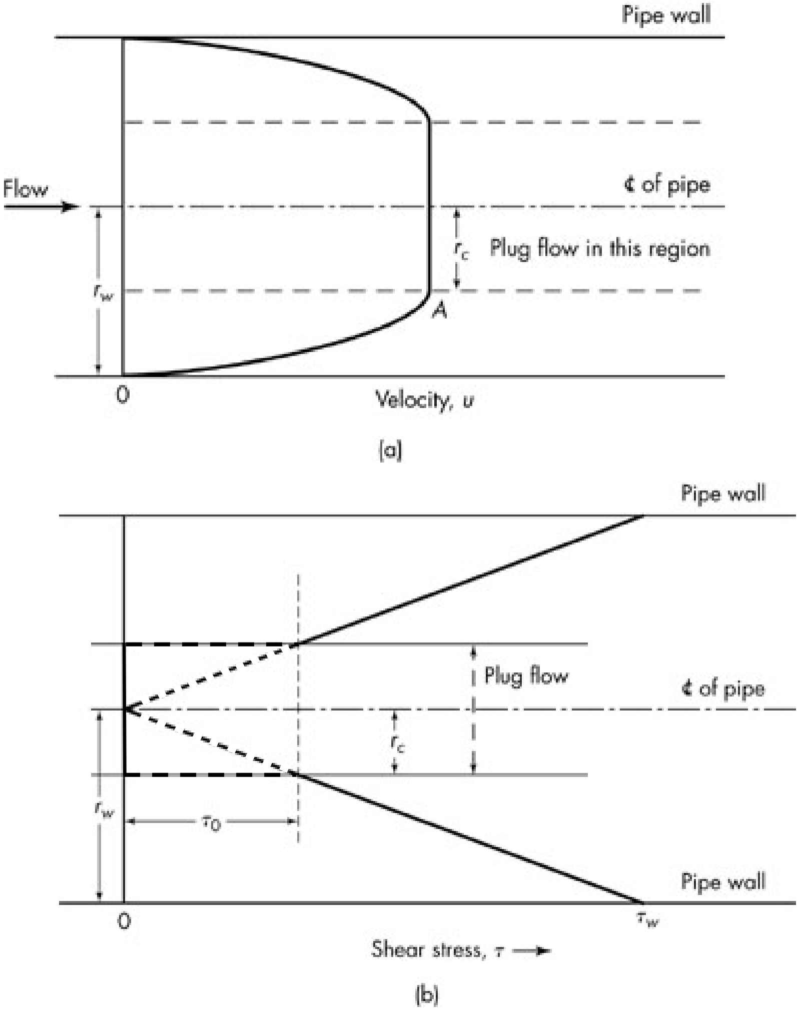
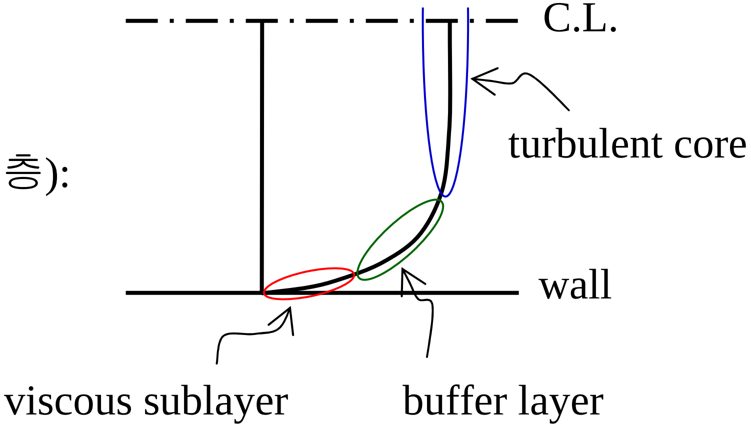


Fig. 5.5. (a) Velocity profile and
(b) Shear diagram
for Bingham plastic flow

- Some non-Newtonian mixtures at high shear violate the zero-velocity (**no-slip**) b. c.
ex) multiphase fluids (suspensions, fiber-filled polymers) → "**slip**" at the wall

Turbulent Flow in Pipes and Channels



viscous sublayer (점성하층):

viscous shear ↑, eddy ×

buffer layer (완충층) or transition layer (전이층):

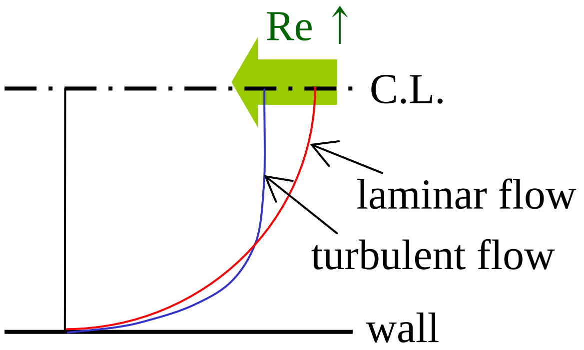
viscous shear & eddy 공존

turbulent core (난류중심부):

viscous shear ↓, eddy diffusion ↑

Velocity profile for turbulent flow:

much flatter than that for laminar flow



Eddies

- in the turbulent core: large but low intensity
- in the buffer layer: small but high intensity

→ Most of the kinetic-energy content of the eddies lies in the buffer zone.

* Velocity distribution for turbulent flow

In terms of dimensionless parameters

$u^* \equiv \bar{V}\sqrt{\frac{f}{2}} = \sqrt{\frac{\tau_w}{\rho}}$: friction velocity

$u^+ \equiv \frac{u}{u^*}$: velocity quotient (무차원)

$y^+ \equiv \frac{yu^*\rho}{\mu} = \frac{y}{\mu}\sqrt{\tau_w \rho}$: distance (무차원) y : distance from tube wall

$(\therefore\ r_w = r + y)$

→ Re based on u^* & y

* Universal velocity distribution equations

i) viscous sublayer: $u^+ = y^+$

ii) buffer layer: $u^+ = 5.00 \ln y^+ - 3.05$

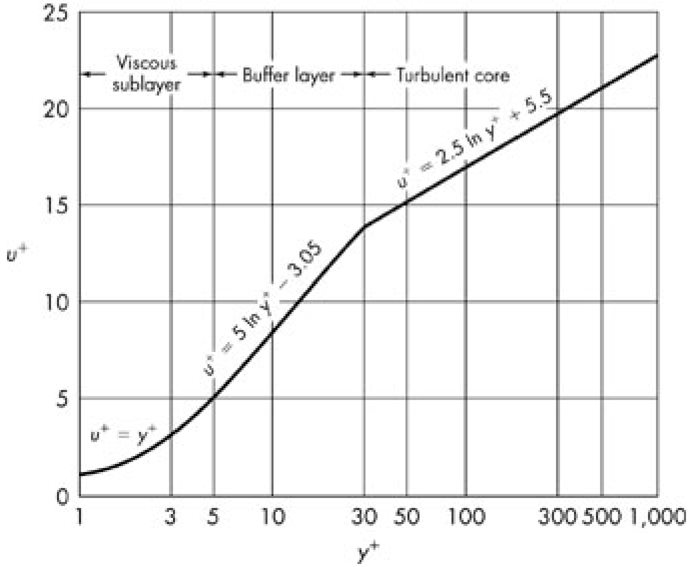
iii) turbulent core: $u^+ = 2.5 \ln y^+ + 5.5$

→ intersection으로부터

$y^+ < 5$ for viscous sublayer

$5 < y^+ < 30$ for buffer zone

$y^+ > 30$ for turbulent core



→ Re > 10,000 이상에서 적용 가능

* Relations between maximum velocity $u_{\max}$ & average velocity $\bar{V}$

For laminar flow, $\bar{V}/u_{\max}$ is exactly 0.5.

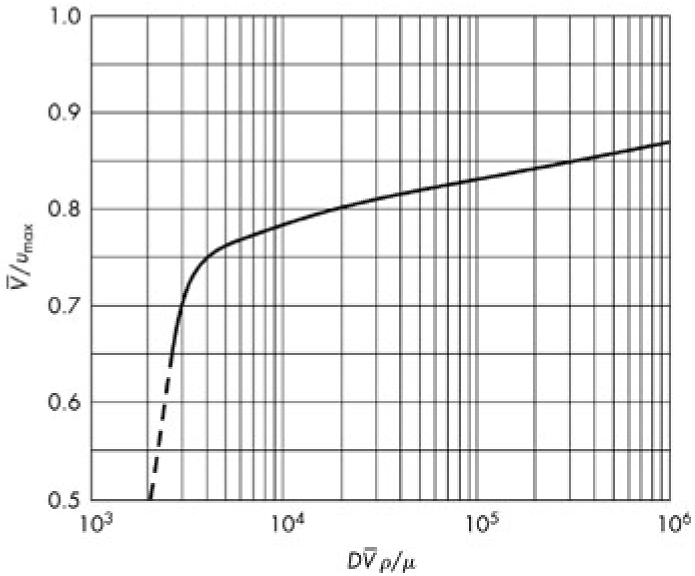
← from Eq. (5.19)

When laminar flow changes to turbulent,

the ratio $\bar{V}/u_{\max}$ changes rapidly

from 0.5 to about 0.7,

& increases gradually to 0.87 when Re=10^6.



* Effect of roughness

Rough pipe → larger friction factor

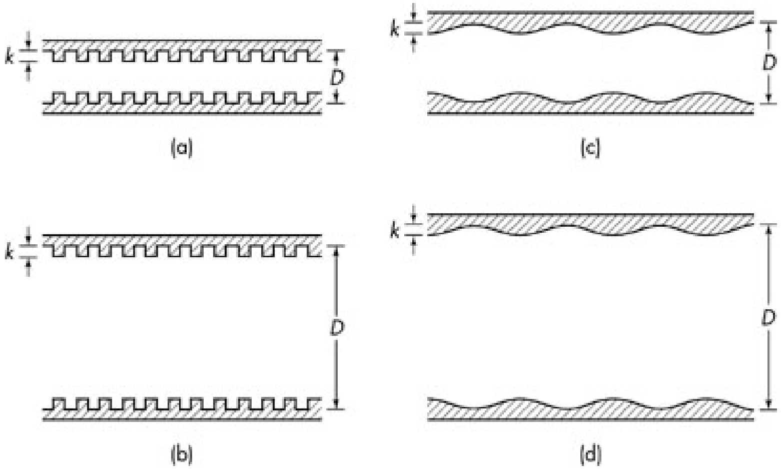
$$\therefore\ f = \text{ft'n of Re} \ \& \ k/D$$

← k : roughness parameter

k/D : relative roughness

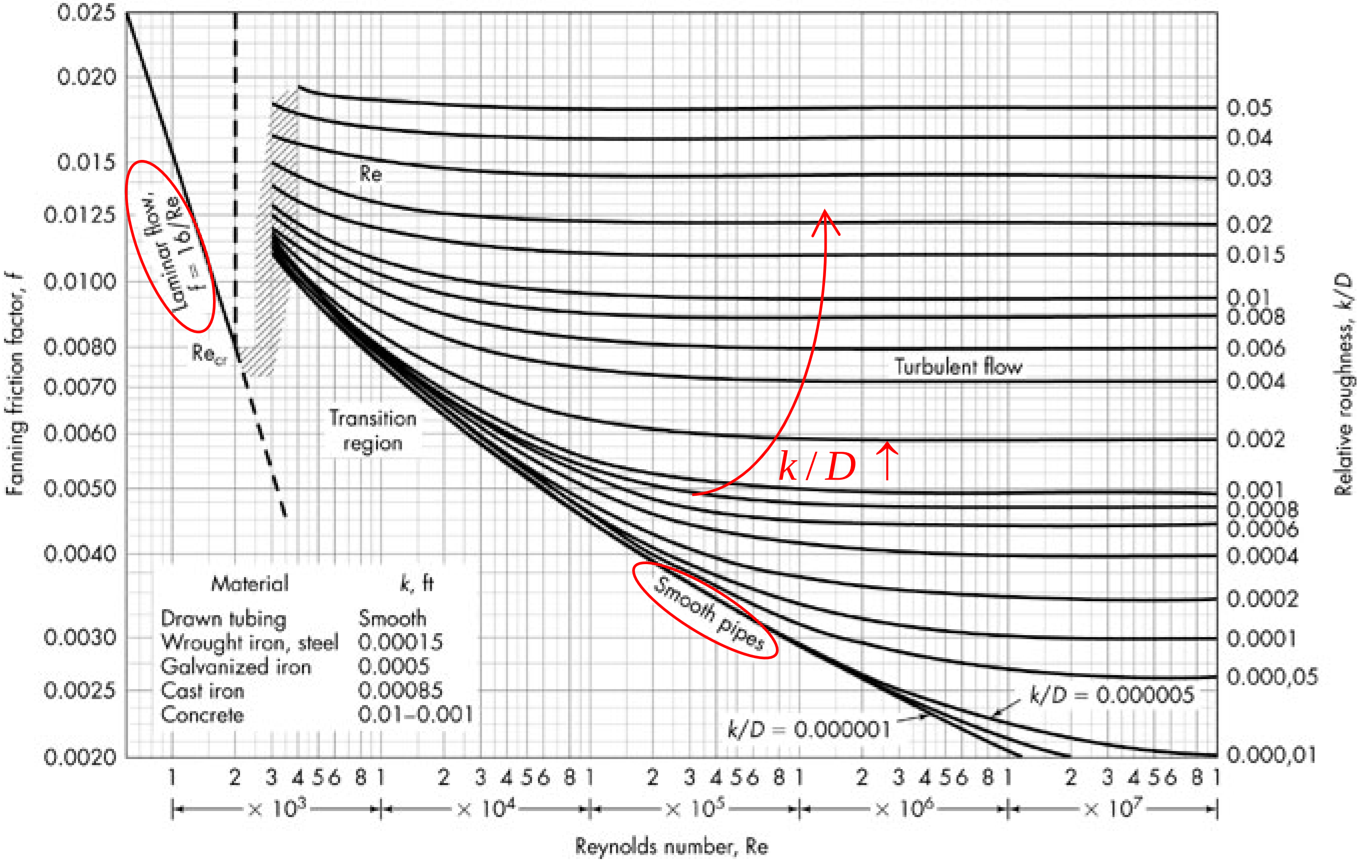
For laminar flow, roughness has no effect

on f unless k is so large.



Types of roughness

* Friction factor chart



Material	k, ft
Drawn tubing	Smooth
Wrought iron, steel	0.00015
Galvanized iron	0.0005
Cast iron	0.00085
Concrete	0.01–0.001

Friction factor plot for circular pipes (log-log plot)

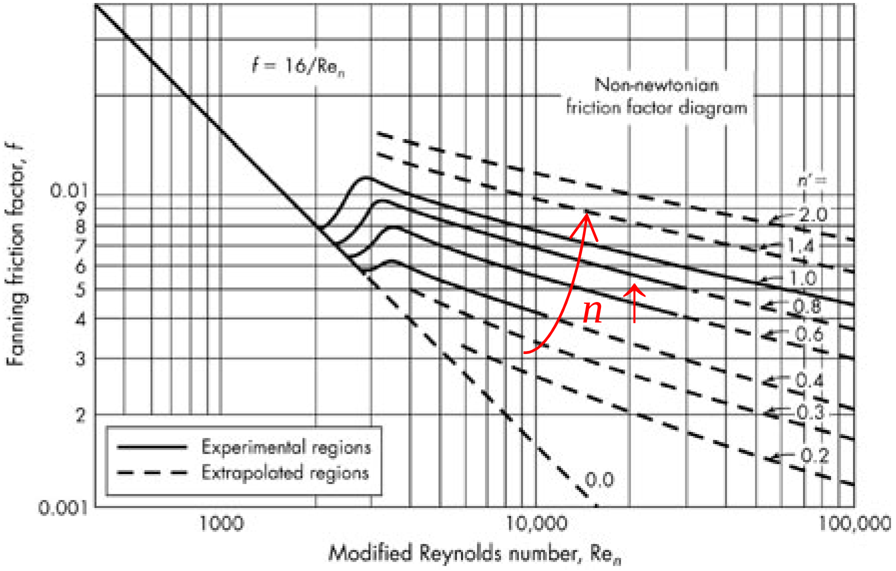
* Friction factor for smooth tube

$$f = 0.046\,\mathrm{Re}^{-0.2} \quad \text{for } 50{,}000 < \mathrm{Re} < 10^6$$

$$f = 0.014 + \frac{0.125}{\mathrm{Re}^{0.32}} \quad \text{for } 3{,}000 < \mathrm{Re} < 3\times10^6$$

(wide range)

* Non-Newtonian fluids

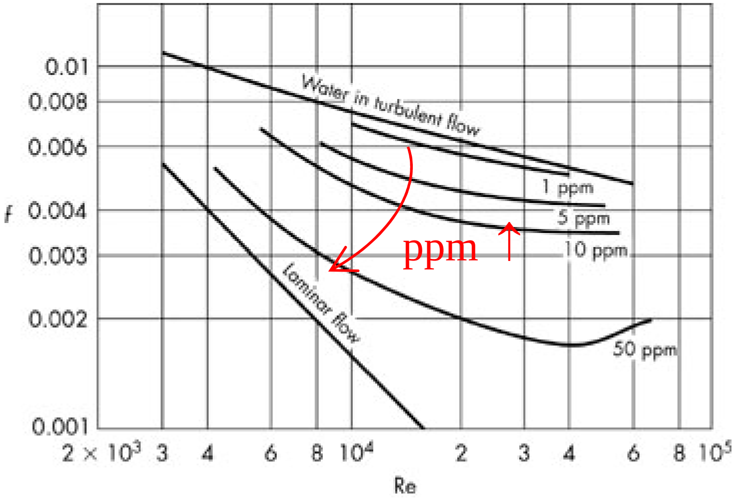


* Drag reduction

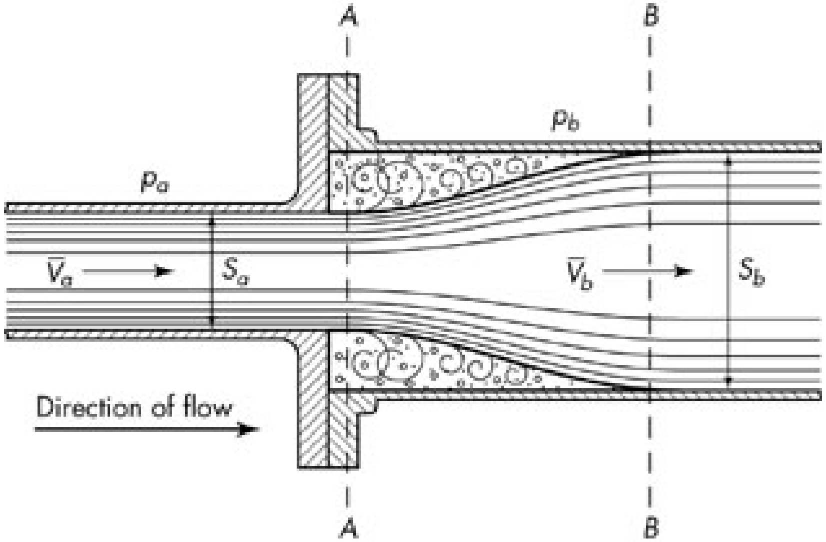
Dilute polymer solutions in water

→ drag reduction in turbulent flow

Application: fire hose (a few ppm of PEO in water can double the capacity of a fire hose)



* Friction loss from sudden expansion



$$h_{fe} = K_e \frac{\bar{V}_a^{\,2}}{2}$$

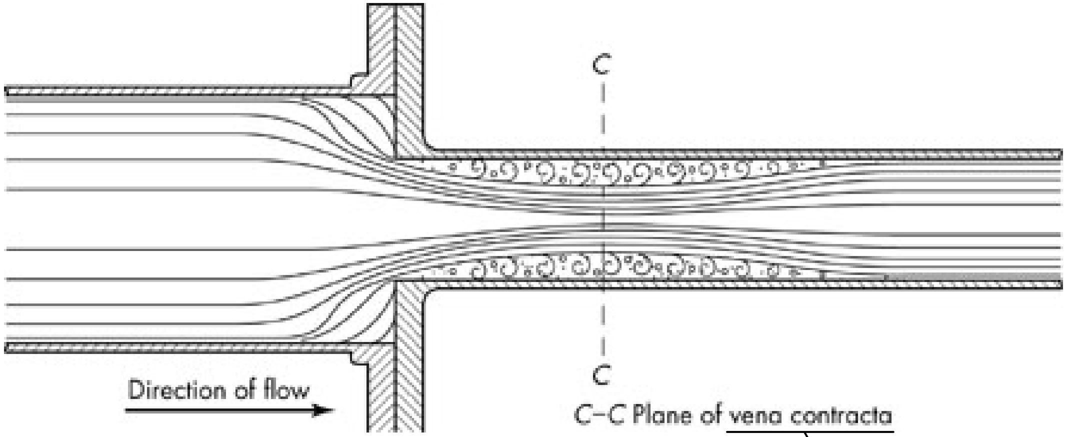
$\bar{V}_a$: average velocity of smaller or upstream section)
K_e : expansion loss coefficient

K_e can be calculated theoretically from the momentum balance equation (4.51) and the Bernoulli equation (4.71).

$$K_e = \left(1 - \frac{S_a}{S_b}\right)^2 \quad \text{for turbulent flow } (\alpha \cong 1 \;\&\; \beta \cong 1)$$

Laminar flow인 경우에는 $\alpha = 2$ & $\beta = 4/3$을 사용하면 K_e 를 구할 수 있다.

* Friction loss from sudden contraction



cross section of minimum area

$$h_{fc} = K_c \frac{\overline{V}_b^{\,2}}{2}$$

$\overline{V}_b$: average velocity of smaller or downstream section)
K_c : contraction loss coefficient

$K_c < 0.1$ for laminar flow → h_{fc} is negligible.

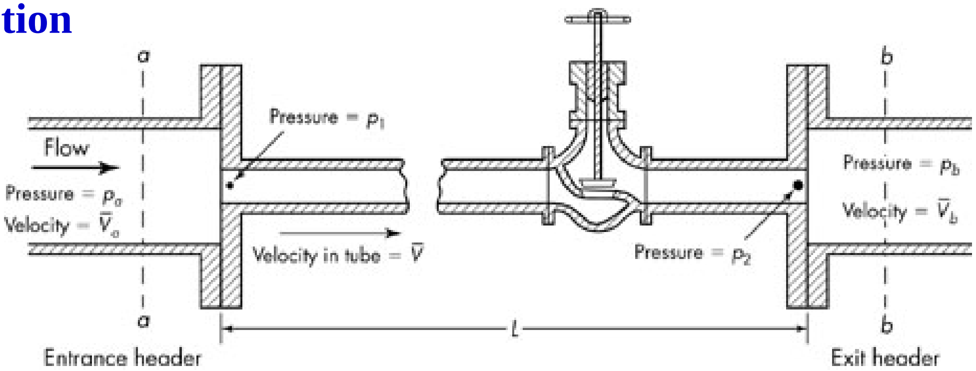
$$K_c = 0.4\left(1 - \frac{S_b}{S_a}\right)$$ for turbulent flow (empirical equation)

* Friction loss from fittings

$$h_{ff} = K_f \frac{\bar{V}_a^{\,2}}{2} \qquad \begin{cases} \bar{V}_a : \text{average velocity in pipe leading to fitting} \\ K_f : \text{fitting loss coefficient} \end{cases}$$

Table 5.1 → Loss coefficients for standard pipe fittings

* Total friction



$$h_f = \left(\underbrace{4f\frac{L}{D}}_{\text{skin friction loss coeff.}} + \underbrace{K_c}_{\text{contraction loss coeff.}} + \underbrace{K_e}_{\text{expansion loss coeff.}} + \underbrace{K_f}_{\text{fitting loss coeff.}} \right) \frac{\bar{V}^2}{2}$$

대입

→ Bernoulli equation without pump: $\dfrac{p_a - p_b}{\rho} + g(Z_a - Z_b) = h_f$

Ex. 5.2) Homework

* Minimizing expansion and contraction losses

. **Contraction loss** can be nearly eliminated by reducing the cross section gradually.

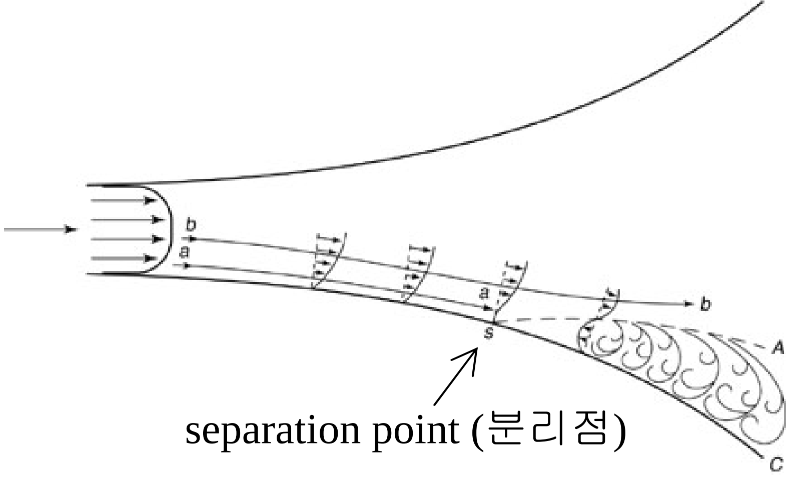
$$\longrightarrow \quad K_c \approx 0.05$$

In this case, separation & *vena contracta* do not occur.

. **Expansion loss** can also be minimized by enlarging the cross section gradually

To minimize expansion loss, the angle between the diverging walls of a conical expander must be less than 7°.

For angles > 35° → The loss through this expander can become greater than that through a sudden expansion.

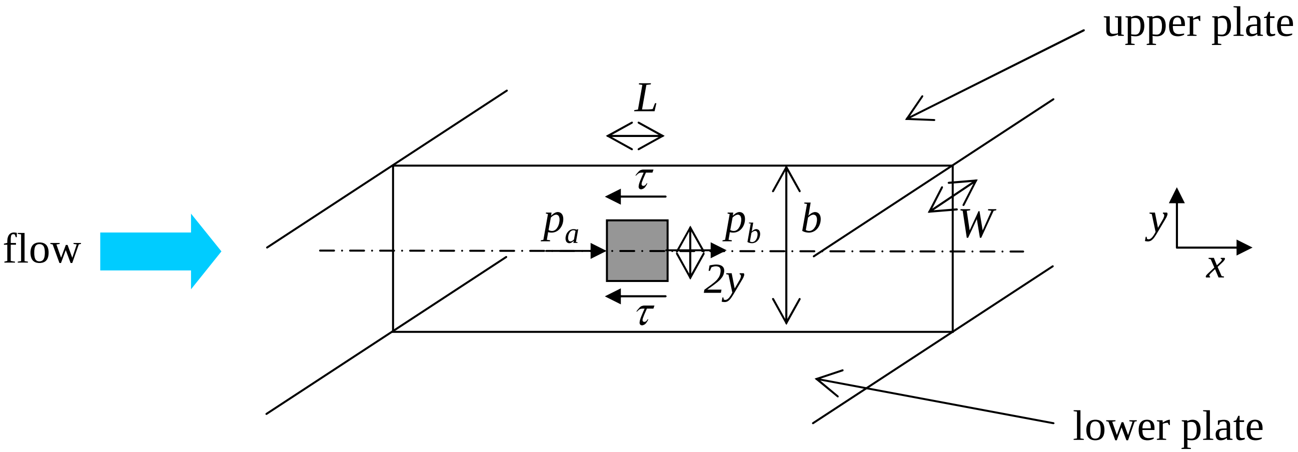


separation point (분리점)

Separation of boundary layer in diverging channel

*** Flow through parallel plates** (Prob. 5.1 & 5.3과 연관)

In laminar flow between infinite parallel plates,



$p_a - p_b = \dfrac{12\mu \bar{V} L}{b^2}$ 임을 보이고 $u/u_{\max}$, $\bar{V}/u_{\max}$를 구하시오.

(풀이) Force balance: $2yWp_a - 2yWp_b = 2\tau LW$ ← from Eq. (4.52)

$$\frac{p_a - p_b}{L} = \frac{\tau}{y} \xleftarrow[\text{대입}]{} \tau = -\mu \frac{du}{dy}$$

$$\frac{(p_a - p_b)}{L\mu}\int_{b/2}^{y} y\,dy = -\int_0^u du$$

적분하면,

$$u = \frac{(p_a - p_b)}{2\mu L}\left(\left(\frac{b}{2}\right)^2 - y^2\right) \xrightarrow[(u_{\max}\text{ at } y=0)]{\text{b.c. 대입}} \therefore\ u_{\max} = \frac{(p_a - p_b)}{2\mu L}\left(\frac{b}{2}\right)^2$$

$$\bar{V} = \frac{1}{S}\int u\,dS = \frac{1}{bW}\int_0^{a/2} uW\,dy$$

$$= \frac{(p_a - p_b)b^2}{12\mu L} \qquad \therefore\ p_a - p_b = \frac{12\mu\bar{V}L}{b^2}$$

$$\therefore\ \frac{u}{u_{\max}} = \left(1 - \left(\frac{y}{(b/2)}\right)^2\right) \qquad \therefore\ \frac{\bar{V}}{u_{\max}} = \frac{2}{3}$$

Related problems:

(Probs.) 5.4, 5.8, 5.10, 5.12, 5.13, 5.17, 5.20 and 5.21