Conduction Heat transfer: Unsteady state

Chapter Objectives

For solving the situations that

- ***Where temperatures do not change with position.***
- ***In a simple slab geometry where temperature vary also with position.***
- ***Near the surface of a large body (semi-infinite region)***

Keywords

- ***Internal resistance***
- ***External resistance***
- ***Biot number***
- ***Lumped parameter anaysis***
- ***1D and multi-dimensional heat conduction***
- ***Heisler charts***
- ***Semi-infinite region***

1 Lumped Parameter Analysis

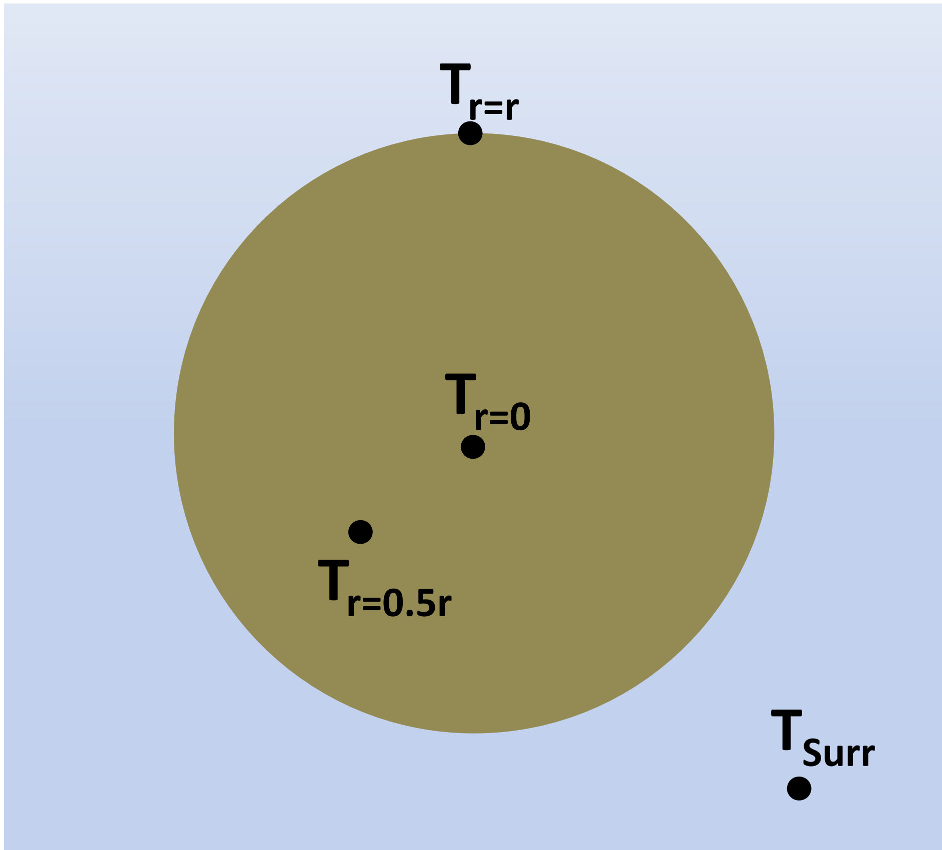

Figure 1. Several temperatures in the system.

In transient, $T_{r=r} = T_{r=0.5r} = T_{r=0}$ **?**

Lumped Parameter Analysis

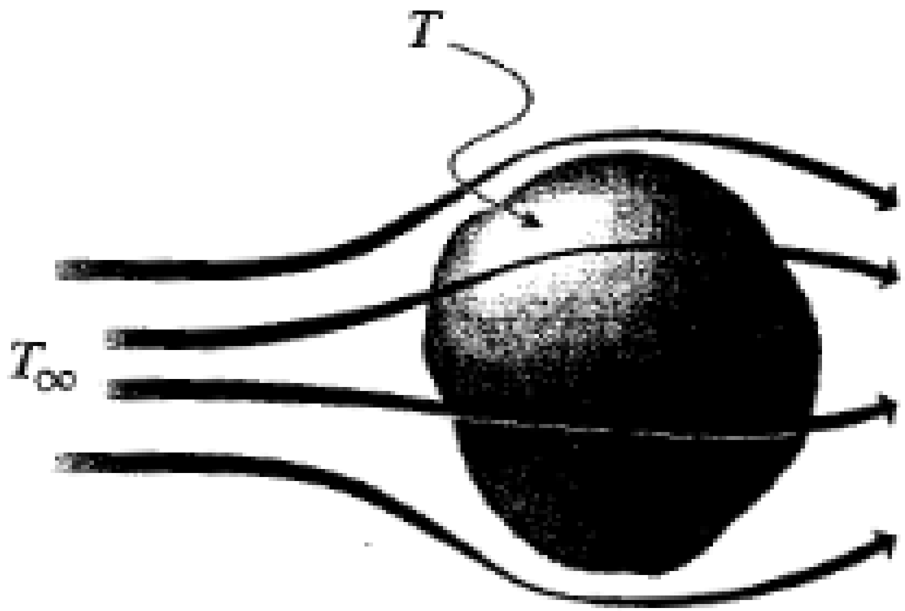

Figure 2. A solid with convection over its surface.

$$-mC_p\Delta T = hA(T - T_\infty)\Delta t \qquad (1)$$

Lumped Parameter Analysis

$$-mC_p \Delta T = hA(T - T_\infty)\Delta\Delta \qquad (1)$$

M : Mass
C_p : Specific heat
h : Convective heat transfer coefficient
A : Surface area
T_∞ : Bulk fluid temperature

$$-\frac{\Delta T}{\Delta t} = \frac{hA}{mc_p}(T - T_\infty)$$

$$\frac{dT}{dt} = -\frac{hA}{mc_p}(T - T_\infty) \qquad (2)$$

Lumped Parameter Analysis

$$T(t=0) = T_i \tag{3}$$

$$\begin{aligned}\theta &= T - T_\infty \\ \theta_i &= \theta(t=0) = T_i - T_\infty\end{aligned}$$

$$\frac{d\theta}{dt} = -\frac{hA}{mc_p}\theta$$

$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = \int_0^t -\frac{hA}{mc_p}dt \tag{4}$$

$$\ln\frac{\theta}{\theta_i} = -\frac{hA}{mc_p}t$$

$$\frac{\theta}{\theta_i} = e^{-\frac{hA}{mc_p}t}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{hA}{mc_p}t\right) = \exp\left(-\frac{t}{\frac{mc_p}{hA}}\right) \tag{5}$$

2 Biot Number

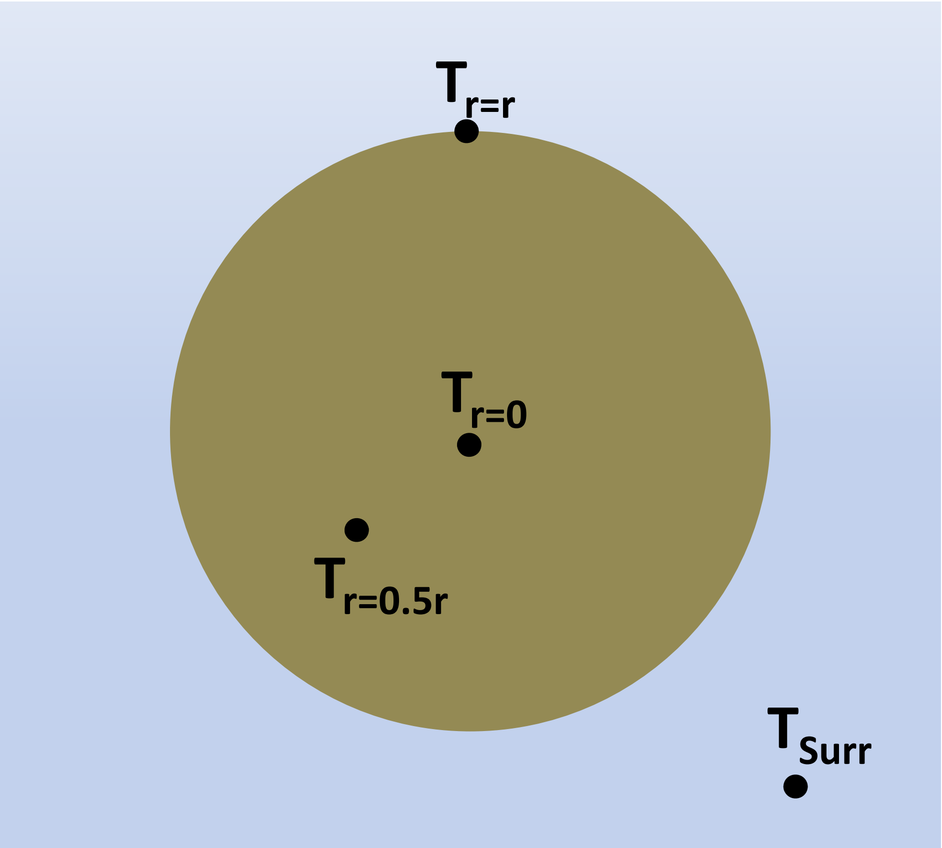

Figure 3. Several temperatures in the system.

So, when can we apply $T_{r=r} = T_{r=0.5r} = T_{r=0}$ **?**

Biot Number

Bi (Biot Number)
: Deciding whether internal resistance can be ignored.

$$Bi\,(Biot\ number) = \frac{hL}{k} = \frac{\frac{L}{kA}}{\frac{1}{hA}} = \frac{conductive\ resistance}{convective\ resistance} \quad (6)$$

$$\frac{h(V/A)}{k} < 0.1 \quad (7)$$

Characteristic Length

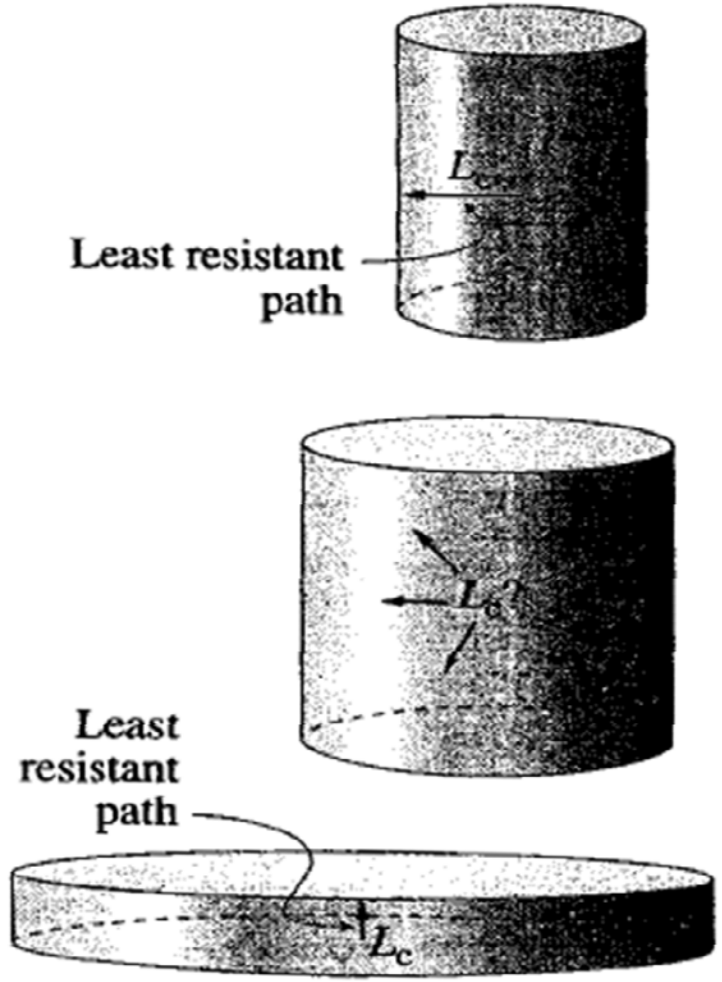

Figure 4. Characteristic lengths for heat conduction in various geometries.

Characteristic length ≡ V/A

Path of least thermal resistance

Characteristic length↓
= Temperature can be changed in short time

$$hL/k < 0.1 \quad (8)$$

Example 1

What is the temperature of the egg after 60min?

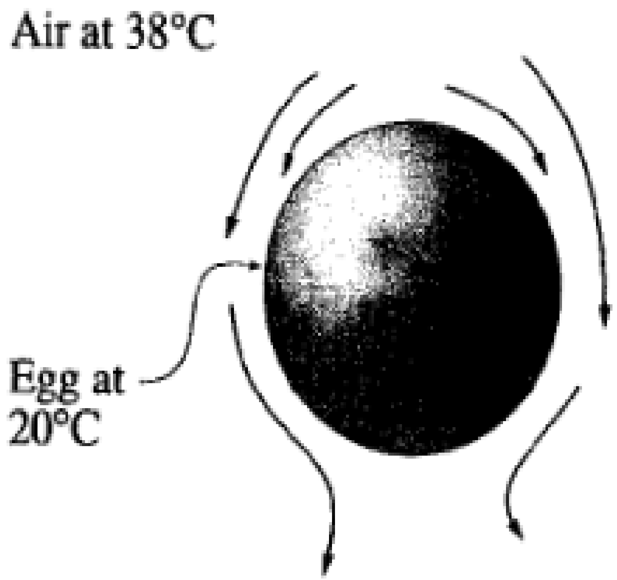

Figure 5. Schematic for Example 1.

Known: *Initial temperature of an egg*

Find: *Temperature of the egg after 60min.*

Given data:

T_i = 20 ℃

T_{air} = 38 ℃

h = 5.2 $W/m^2 \cdot K$

ρ = 1035 kg/m^3

C_p = 3350 $J/kg \cdot K$

k = 0.62 $W/m \cdot K$

Assumption:

1. ***Egg is approximately spherical.***

2. ***Surface heat transfer coefficient provided is an average value.***

3. ***Lumped parameter analysis.***

Bi (Biot Number) = hV / Ak = 0.07 < 0.1

Being Bi <0.1, lumped analysis can be applied!

Using (Eqn. 5),

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{hA}{mc_p}t\right)$$

$$\frac{T-38}{20-38} = \exp\left(-\frac{5.2[W/m^2 \bullet K] \times 0.00785[m^2]}{1035[kg/m^3] \times 60 \times 10^{-6}[m^3] \times 3350[J/Kg \bullet K]} 3600[s]\right)$$

Then, T = 29.1 ℃

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3 When Internal Resistance Is Not Negligible

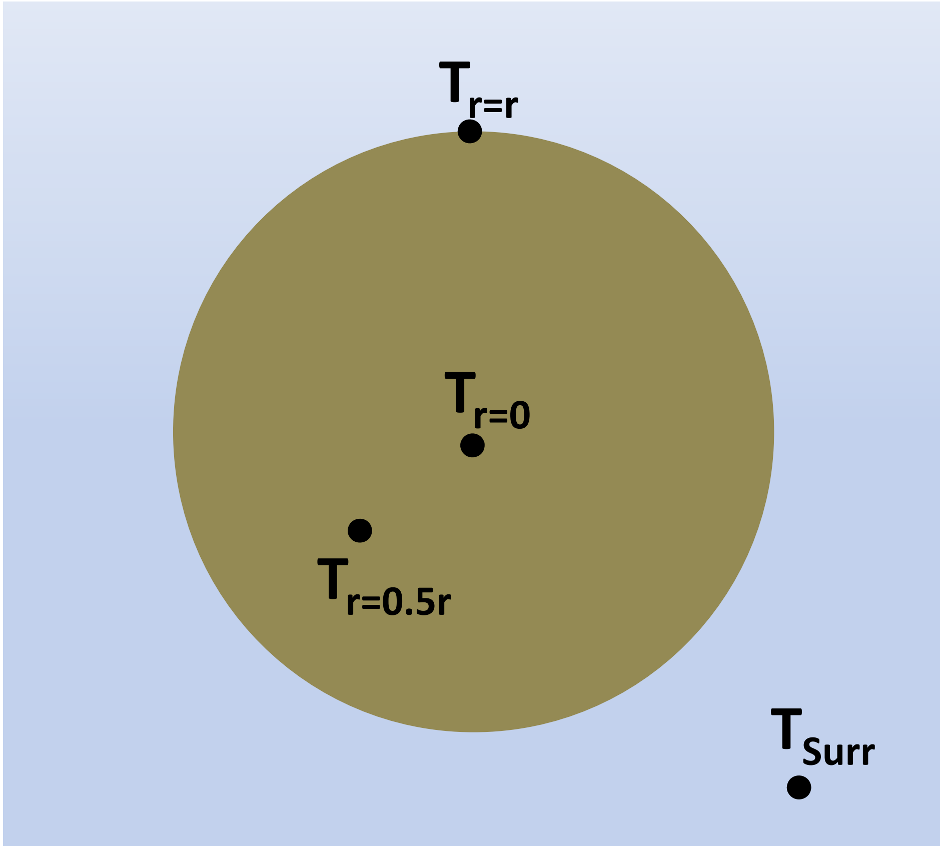

Figure 1: Several temperatures in the system.

The situations, $T_{r=r} \neq T_{r=0.5r} \neq T_{r=0}$ (*i.e.* Bi ≥ 0.1)

When Internal Resistance Is Not Negligible

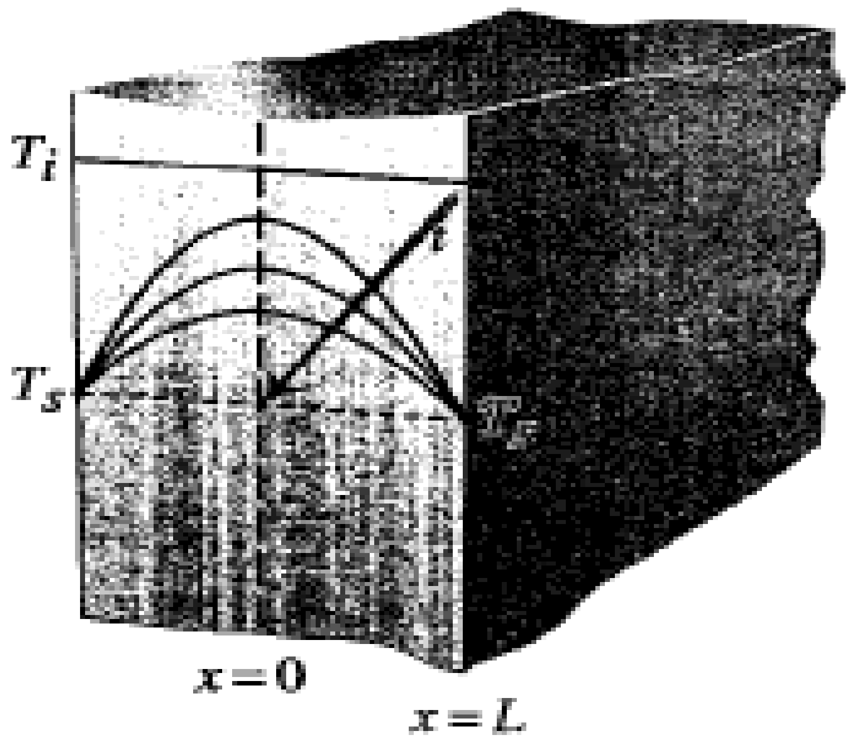

Figure 6. Schematic of a slab showing the line of symmetry at x = 0 and the two surfcaes at x = L and at x = -L maintained at temperature T_S. The material is very large (extends to infinity) in the other two directions.

When Internal Resistance Is Not Negligible

$$\underbrace{\rho c_p \frac{\partial T}{\partial t}}_{\text{storage}} + \underbrace{u \frac{\partial T}{\partial x}}_{\text{no bulk flow}}{}^{0} = \underbrace{k\left(\frac{\partial^2 T}{\partial x^2}\right)}_{\text{conduction}} + \underbrace{Q}_{\substack{\text{no heat} \\ \text{generation}}}{}^{0}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial x^2} \qquad (9)$$

Boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0,t} = 0 \ \ (\text{for } symmetry) \qquad (10)$$

$$T(L, t > 0) = T_s \qquad (11)$$

When Internal Resistance Is Not Negligible

Initial condition

$$T(x, t = 0) = T_i \qquad (12)$$

$$\frac{T - T_s}{T_i - T_s} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)\pi} \cos\frac{(2n+1)\pi x}{2L} e^{-\alpha\left(\frac{(2n+1)\pi}{2L}\right)^2 t} \qquad (13)$$

α (Thermal diffusivity) = $k/\rho C_p$

How Temperature Changes with Time

For visualizing Temperature vs. Position and Time,
infinite series should be simplified

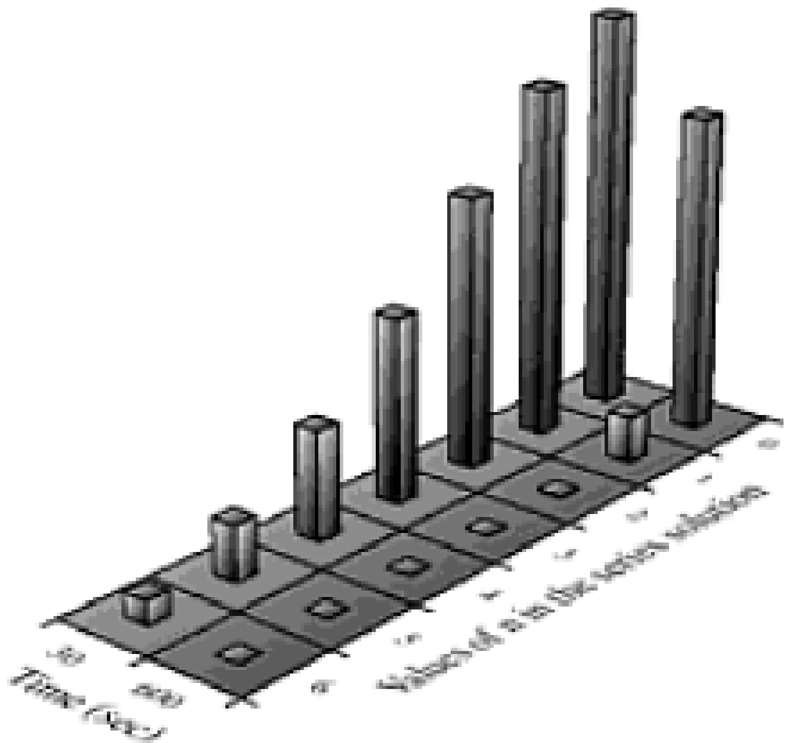

Figure 7. The terms in the series (n = 0, 1, ... in Equation 5.13) drop off rapidly for values of time. Calculations are for F_O = 0.0048 at 30 s and F_O = 0.096 at 600 s for a thickness of L = 0.03 m and a typical α = 1.44 x 10-7m2/s for bio materials.

How Temperature Changes with Time

***Comparing** different **terms** at each time(t= 30s, t= 600s),*

Contribution decays
- ***Gradually** at t= 30s*
- ***Rapidly** at t= 600s*

$$\frac{T - T_s}{T_i - T_s} = \frac{4}{\pi} \cos\frac{\pi x}{2L} e^{-\alpha\left(\frac{\pi}{2L}\right)^2 t} \qquad (15)$$

$$\ln\frac{T - T_s}{T_i - T_s} = \ln\left(\frac{4}{\pi} cos\frac{\pi x}{2L}\right) - \alpha\left(\frac{\pi}{2L}\right)^2 t \qquad (16)$$

Temperature Change with Position and Spatial Average

$$\frac{T - T_s}{T_i - T_s} = \frac{4}{\pi} \cos \frac{\pi x}{2L} e^{-\alpha \left(\frac{\pi}{2L} \right)^2 t} \qquad (15)$$

$$\ln \frac{T - T_s}{T_i - T_s} = \ln\left(\frac{4}{\pi} \cos \frac{\pi x}{2L} \right) - \alpha \left(\frac{\pi}{2L} \right)^2 t \qquad (16)$$

- ***We can see that temperature varies as a cosine function***
- ***Therefore, we need to define spatial average temperature***

Spatial average temperature

$$T_{av} = \frac{1}{L}\int_0^L T dx \tag{17}$$

Applying (5.17) to (5.16) gives

$$\ln\frac{T_{av} - T_s}{T_i - T_s} = \ln\frac{8}{\pi^2} - \alpha\left(\frac{\pi}{2L}\right)^2 t \tag{18}$$

Temperature Change with Size

$$\frac{\alpha t}{L^2} = -\frac{4}{\pi^2} \ln\left[\frac{\pi^2}{8}\left(\frac{T_{av} - T_s}{T_i - T_s}\right)\right] \qquad (19)$$

Charts Developed from the Solutions: Their Uses and Limitations.

$$\frac{T-T_s}{T_i-T_s}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{(2n+1)\pi}\cos\left[\frac{(2n+1)\pi}{2}\frac{x}{L}\right]e^{-\left(\frac{(2n+1)\pi}{2}\right)^2\frac{\alpha t}{L^2}} \qquad (20)$$

- ***It can be seen that temperature is a function of x/L and $\alpha t/L^2$***

- ***Charts are developed because of the complexity of the calculation of series.***

- ***Charts are developed with the condition of n=0. In other words, it is a plot of Eqn. 5 And it is also called Heisler chart.***

- ***There are some assumptions for the development of the charts. These are:***

1. *Uniform initial temperature*
2. *Constant boundary fluid temperature*
3. *Perfect slab, cylinder or sphere*
4. *Far from edges*
5. *No heat generation (Q=0)*
6. *Constant thermal properties (k, α, c_p are constants)*
7. *Typically for times long after initial times, given by $\alpha t/L^2>0.2$*

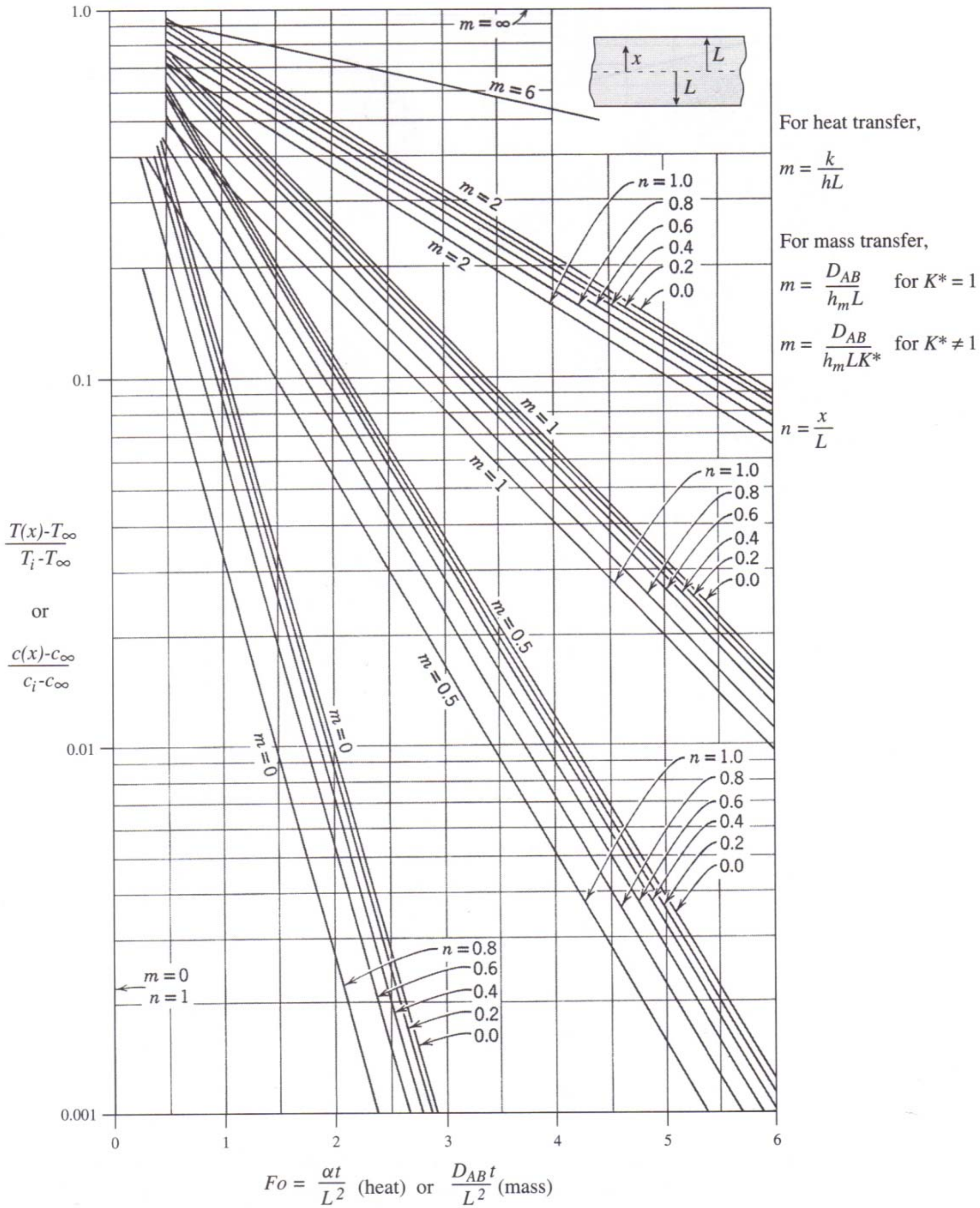

Figure 8. Unsteady state diffusion in a large slab

Example 2. Temperatures Reached During Food Sterilization

Figure 9. A cylindrical can containing food to be sterilized.

- Surface temperature of a slab of tuna is suddenly increased

- Find the temperature at the center of the slab after 30 min

- Given data:

1. Thickness of slab = 25 mm
2. Thermal diffusivity of the slab, $\alpha = 2 \times 10^{-7} m^2 / s$
3. Initial temperature = 40℃
4. Surface temperature = 121℃
5. Time of heating = 1800s

- Assumptions

1. Heating from the side is ignored
2. Thermal diffusivity is constant

$$n = \frac{x}{L} = \frac{0}{0.0125} = 0$$

$$m = \frac{k}{hL} = 0$$

$$F_0 = \frac{\alpha t}{L^2} = \frac{2 \times 10^{-7}\left[m^2/s\right]1800[s]}{(0.0125)^2\left[m^2\right]} = 2.3$$

$$\frac{T - T_\infty}{T_i - T_\infty} = 0.0043$$

So the temperature T = 120.65℃ after 30 minutes of heating

Convective Boundary Condition

- *We have considered a negligible external fluid resistance to heat transfer.*

- *But if we consider external fluid resistance in addition to internal fluid resistance,*

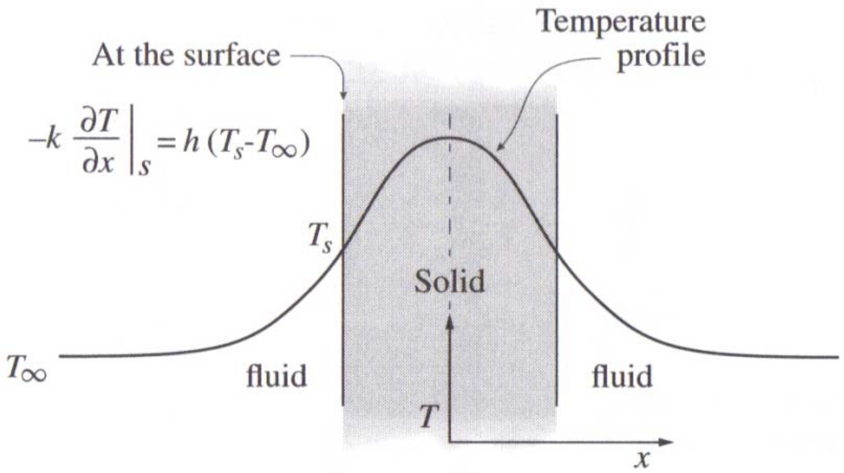

Figure 10. In convective boundary condition, surface temperature is not the same as the bulk fluid temperature, T_∞, signifying additional fluid resistance .

- *At the surface,*

$$-k\frac{\partial T}{\partial x}\bigg|_s = h\left(T_s - T_\infty\right)$$

The solution is generalized form of Eqn. 5.13 and you can refer to Heisler chart as well.

Numerical Methods as Alternatives to the Charts

- *In practice, however, such conditions dealt with above are not that simple*

- *Limitations of the analytical solutions can be overcome using numerical, computer-based solutions*

4 Transient Heat Transfer in a Finite Geometry-Multi-Dimensional Problems

• *We should consider the situation two- and three-dimensional effect yields*

• *A finite geometry is considered as the intersection of two or three infinite geometries*

$$\frac{T_{xyz,t}-T_s}{T_i-T_s}=\left(\frac{T_{x,t}-T_s}{T_i-T_s}\right)_{\substack{\inf inite \\ x\, slab}}\left(\frac{T_{y,t}-T_s}{T_i-T_s}\right)_{\substack{\inf inite \\ y\, slab}}\left(\frac{T_{z,t}-T_s}{T_i-T_s}\right)_{\substack{\inf inite \\ z\, slab}} \tag{21}$$

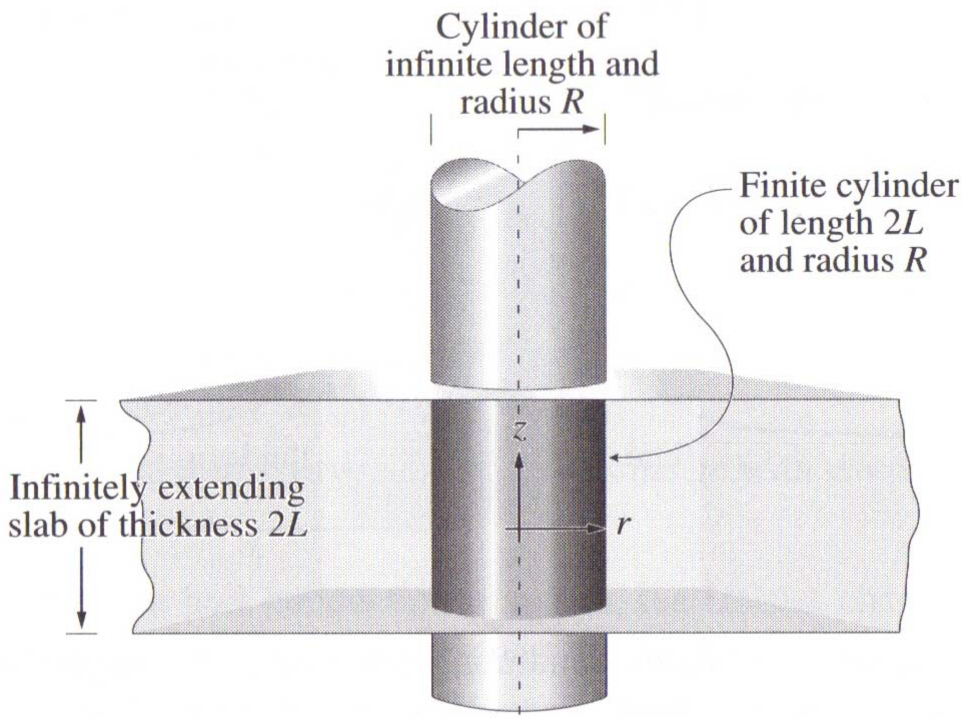

Figure 11. A finite cylinder can be considered as an intersec tion of an infinite cylinder and a slab

$$\frac{T_{r,z,t}-T_s}{T_i-T_s}=\left(\frac{T_{r,t}-T_s}{T_i-T_s}\right)_{\substack{\inf inite\\ cylinder}}\left(\frac{T_{z,t}-T_s}{T_i-T_s}\right)_{\substack{\inf inite\\ slab}} \tag{22}$$

5 Transient Heat Transfer in a Semi-infinite Region

- *A semi-infinite region extends to infinity in two directions and a single identifiable surface in the other direction*

- *You can see Fig. 5.11 extends to infinity in the y and z directions and has an identifiable surface at x=0*

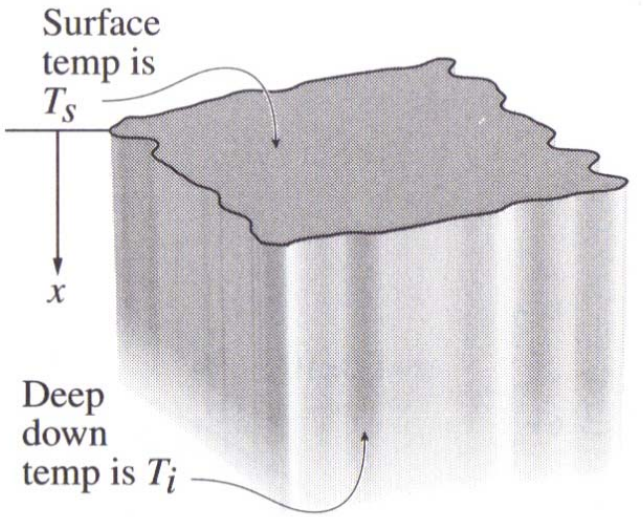

Figure 12. Schematic of a semi-infinite region showing only one identifiable surface.

• *It can be used practically in heat transfer for a relatively short time and/or in a relatively thick material*

• *The governing equation with no bulk flow and no heat generation is*

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad (23)$$

• *The boundary conditions are*

$$T(t = 0) = T_i \qquad (24)$$

$$T(x = 0) = T_s \qquad (25)$$

•*The initial condition is*

$$T(x \to \infty) = T_i \qquad (26)$$

- *The solution is*

$$\frac{T - T_i}{T_s - T_i} = 1 - erf\left[\frac{x}{2\sqrt{\alpha t}}\right] \qquad (27)$$

The function erf(η) is called error function and given by

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

And here, $\eta = \dfrac{x}{2\sqrt{\alpha t}}$

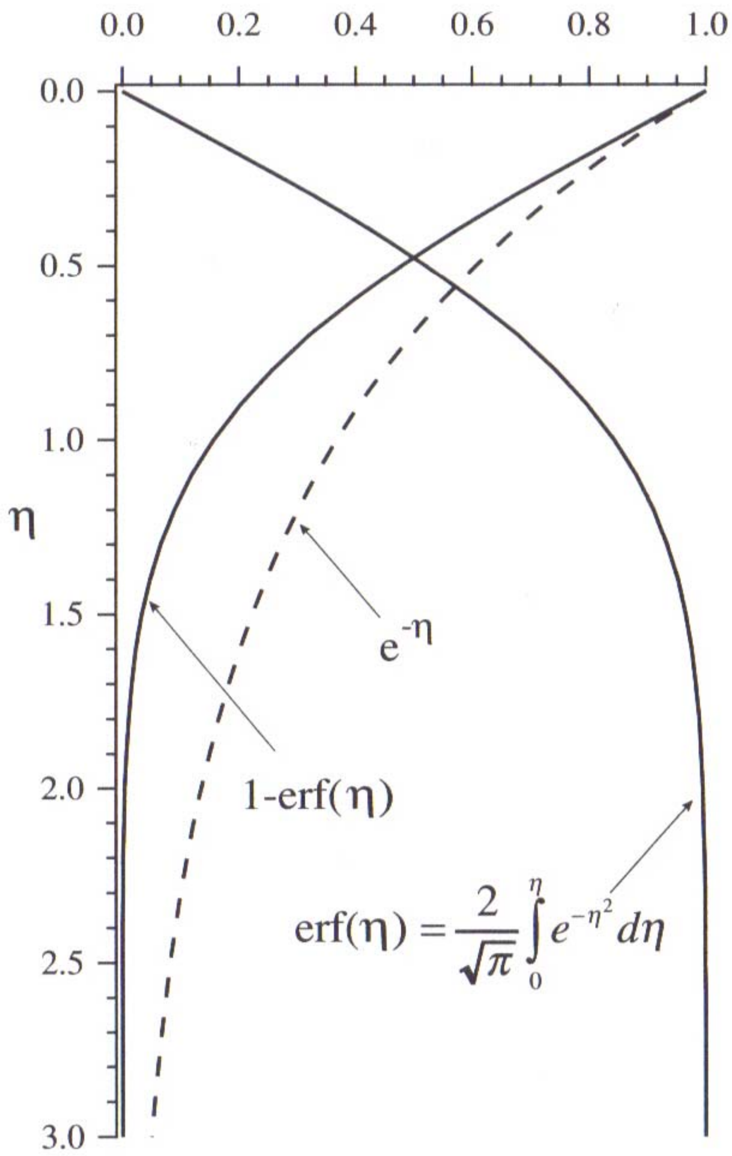

Figure 13. Comparison of the complementary error function (1-erf(η)) with an exponential $e^{-\eta}$

• *Heat flux at the surface of the semi-infinite region can be calculated with chain rule*

$$q_s'' = -k\frac{dT}{dx}\bigg|_{x=0} = -k\frac{dT}{d\eta}\frac{d\eta}{dx}\bigg|_{x=0}$$

$$= -k\left(T_s - T_i\right)\left(-\frac{2}{\sqrt{\pi}}e^{-\eta^2}\right)_{\eta=0}\frac{1}{2\sqrt{\alpha t}}$$

$$= \frac{k\left(T_s - T_i\right)}{\sqrt{\pi\alpha t}} \qquad (28)$$

- *The situation we can approximate semi-infinite region*

$$\frac{x}{2\sqrt{\alpha t}} \geq 2$$

$$x \geq 4\sqrt{\alpha t} \qquad (29)$$

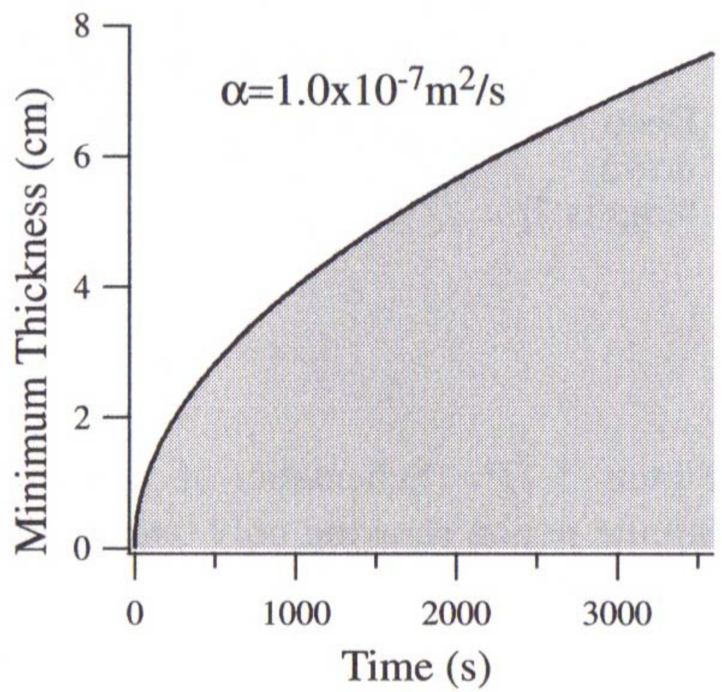

Figure 14. Plot of Eqn. 29, illustrating the minimum thickness of a material for which error function solution can be used.

• *Other boundary conditions*

1. Convective boundary condition

$$-k\frac{\partial T}{\partial x}\bigg|_{surface} = h\left(T_{surface} - T_{\infty}\right)$$

The solution is

$$\frac{T - T_i}{T_{\infty} - T_i} = 1 - erf\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{\frac{hx}{k}+\frac{h^2\alpha t}{k^2}}\left(1 - erf\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right) \qquad (30)$$

2. Specified surface heat flux boundary condition

$$q''_{surface} = q''_s \tag{31}$$

The solution is

$$T - T_i = \frac{2}{k} q''_s \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{q''_s x}{k}\left(1 - erf\left(\frac{x}{2\sqrt{\alpha t}}\right)\right) \tag{32}$$

Example 3 Analysis of Skin Burns

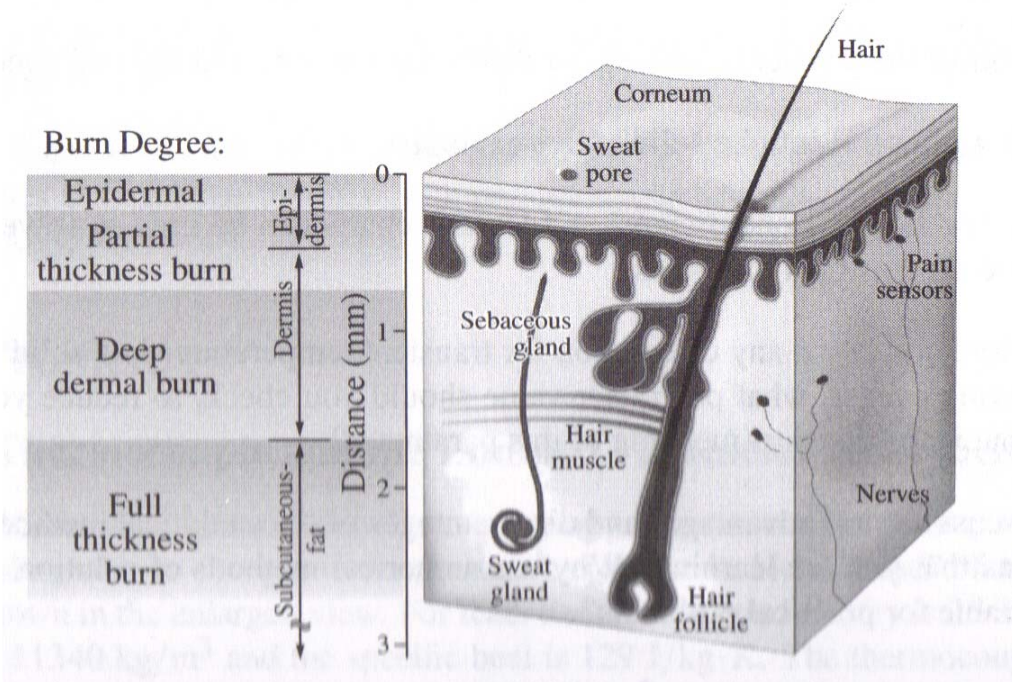

Figure 15. Section of a skin with degrees of burn superimposed on it.

- *A thermal burn occurs as a result of an elevation in tissue temperature above a threshold value for a finite period of time*
- *The intensity of thermal burn is divided into four degrees*

6 Chapter Summary-Transient Heat Conduction

- *No Internal Resistance, Lumped Parameter*

1. *The thermal resistance of the solid can be ignored if a Biot number is less than 0.1.*

2. *As thermal resistances are ignored, temperature is a function of time only.*

- *Internal Resistance is Significant*

1. *When internal resistance is significant (Bi>0.1), temperature is a function of both position and time*

2. *For an infinite slab, infinite cylinder and spherical geometry, the solutions are given as Heisler chart. You can find it on pages 327~329.*

3. *For finite slab and finite cylinder, the solutions are intersection of the infinite slabs and cylinder.*

4. *Materials with thickness* $L \geq 4\sqrt{\alpha t}$ *are considered effectively semi-infinite*