2.4.6 MODEL CONDITIONING

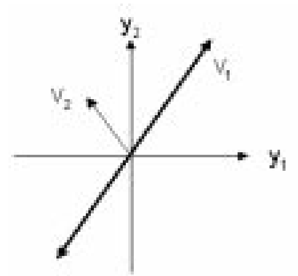
Ill-Conditioned Process ?

- Consider the following process:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} u_1 + \begin{bmatrix} 4 \\ 6 \end{bmatrix} u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} (u_1 + 2u_2)$$

Two column vectors of the steady state gain matrix are colinear. As a result, $[y_1\ y_2]^T$ lies on $\mathbf{v}_1$ for any u_1 and u_2.

If the set point is given outside $\mathbf{v}_1$, it can never be attained.



- This time, let the steady state gain matrix be

$$\begin{bmatrix} 2 & 4 \\ 3 & 6.2 \end{bmatrix}$$

Two column vectors are nearly colinear.
Assume that $\mathbf{y}^{sp} = (-2,\ 3)^T \perp \mathbf{v}_1$.
The input is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 6.2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -61 \\ 30 \end{bmatrix}$$

On the other hand, for $\mathbf{y}^{sp} = (2\ 3)^T = \mathbf{v}_1$, the input is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

– It is possible to control $\mathbf{y}$ along the $\mathbf{v}_2$ direction but a large input possibly beyond the constraints is required.

– Does it make sense to try to control both y_1 and y_2 independently ? The answer will depend on the requirements of the process, but in many cases would be ‘Not !”

– If we give up one of the outputs, say y_2, and control only y_1 at y_1^{sp}, only a small control energy will be required. In this case, y_2 will stay at around $1.5y_1^{sp}$.

- Since the above gain matrix has a very large condition number, we say that the process is *ill-conditioned* or has a strong directionality.

Analysis using SVD

- Let

$$\mathbf{G} = [\ \mathbf{W}_1\ \mathbf{W}_2\ ] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^T \\ \mathbf{V}_2^T \end{bmatrix}$$

- Assume that $\Sigma_1 \gg \Sigma_2$ where $\dim(\Sigma_1) = m < n$,

$$\mathbf{y} = \mathbf{G}\mathbf{u} \ \Rightarrow\ \mathbf{y} \approx \mathbf{W}_1 \Sigma_1 \mathbf{V}_1^T \mathbf{u}$$

  The output is dominated by the modes with large singular values.

- On the other hand,

$$\mathbf{u} = \mathbf{G}^+\mathbf{y} = [\ \mathbf{V}_1\ \mathbf{V}_2\ ] \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{W}_1^T \\ \mathbf{W}_2^T \end{bmatrix} \ \Rightarrow\ \mathbf{u} \approx \mathbf{V}_2 \Sigma_2^{-1} \mathbf{W}_2^T \mathbf{y}$$

  where $^+$ denotes the pseudo-inverse. The input is mostly determined by less significant modes associated with small singular values.

- SVD can be extended to a dynamic gain.

$$\mathbf{G}(j\omega) = \mathbf{W}(j\omega)\Sigma(j\omega)\mathbf{V}^T(j\omega)$$

Model Conditioning in Commercial Packages

step 0 It is assumed that operating regime for output $\mathbf{y}$ is given with priority for each output.

step 1 From the SVD of $\mathbf{G}$(steady state gain matrix), count the number of significant modes. Let it be m. Notify that $n - m$ outputs are better to be removed from the controlled variables (CVs).

step 2 Using $\mathbf{u} = \mathbf{G}^{+}\mathbf{y}$, check if the input constraints can be violated for any $\mathbf{y}$ within the defined set. If unacceptable, do the next step.

step 3 The designer takes out some of low propority outputs from CVs. Let the reduced input-output model be $\mathbf{y}_r = \mathbf{G}_r\mathbf{u}$. Repeat step 2 for the reduced model until the estimated input is acceptable for all possible $\mathbf{y}_r$.

This idea can be slightly generalized to include quantitative weighting for each output (rather than strict priority).

Model conditioning is needed not only to prevent input constraint violation (which would be automatically handled by the constrained MPC), but because low-gain directions are very difficult to identify and gains typically carry large multiplicative errors (sometimes more than 100%).

2.4.7 BLOCKING

- Consider an MPC problem with $m = 30$, $n_u = 4$. At every sampling instance, MPC has to decide 120 variables through QP. Are all these values truly impotant in the prediction of major output modes ?
- The technique to reduce the number of input decision variables while minimizing the degradation of the intended control performance is called *blocking*.
- Blocking can enhance robustness of MPC, too.

Concept of Blocking

- Blocking is an approximation of the future input values by a linear combination of appropriately chosen small number of basis vectors.

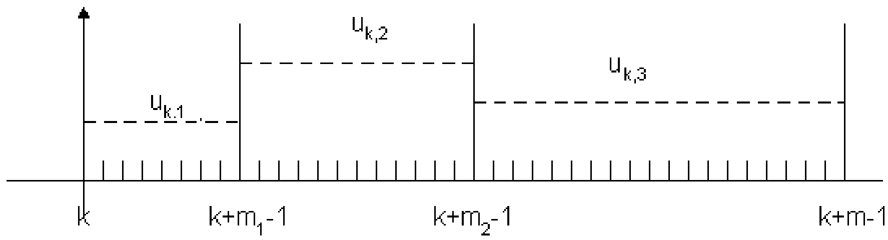
$$\Delta \mathcal{U}_k \approx \mathbf{b}_1 \Delta u_{1k}^* + \mathbf{b}_b \Delta u_{bk}^* = \mathbf{B} \Delta \mathcal{U}_k^* \qquad b \ll m \times n_u$$

  where $m \times n_u$ is the dimension of $\mathcal{U}_k$.
  $\mathbf{B}$ is called a *blocking matrix*.

- QP determines $\Delta \mathcal{U}_k^*$ instead of $\Delta \mathcal{U}_k$.
- The most important step in blocking is to choose an appropriate blocking matrix.

Time-Domain Blocking - Signal Approximation

Divide the control horizon into several subintervals and decide piecewise constant input value for each subinterval.



$$\underbrace{\begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+m_1-1} \\ \Delta u_{k+m_1} \\ \Delta u_{k+m_1+1} \\ \vdots \\ \Delta u_{k+m_2-1} \\ \Delta u_{k+m_2} \\ \Delta u_{k+m_2+1} \\ \vdots \\ \Delta u_{k+m-1} \end{bmatrix}}_{\Delta \mathcal{U}_k} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \Delta u_{k,1} \\ \Delta u_{k,3} \\ \Delta u_{k,3} \end{bmatrix}}_{\Delta \mathcal{U}_k^*}$$

Rather heuristic.

Many industrial algorithms employ the above technique.

SVD-based Blocking

SVD of the pulse reponse matrix informs us which input directions excite the significant output directions.
Let the SVD of the truncated pulse response matrix over the control and prediction horizons be

$$\mathbf{H} = [\ W_1\ W_2\ ] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

where

$$\mathbf{H} = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_m & h_{m-1} & \cdots & h_1 \\ \vdots & \vdots & \vdots & \vdots \\ h_p & h_{p-1} & \cdots & h_{p-m+1} \end{bmatrix}$$

If $\Sigma_1 \gg \Sigma_2$, we can choose

$$\mathbf{B} = V_1$$

$\Rightarrow$ Approximate $\Delta\mathcal{U}_k$ as a linear combination of dominant input principal vectors.

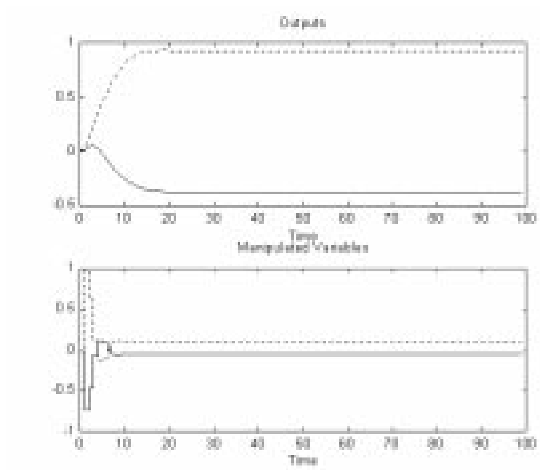
- considered to be better than the time-domain blocking in that it provides structural approximation of MIMO systems, too.

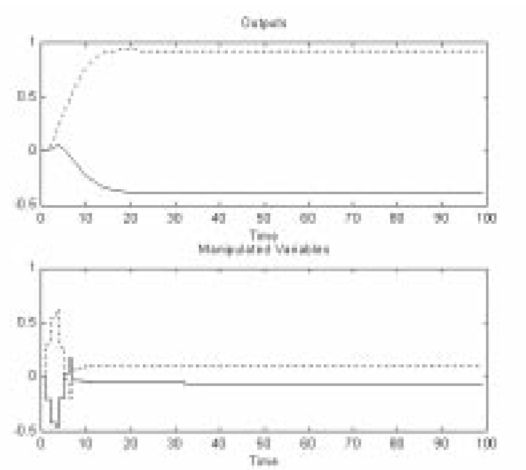
[Ex. 1] No Model Error Case

$$G(s) = G^{model}(s) = \begin{bmatrix} \frac{17}{60.48s^2+15.6s+1} & \frac{5}{19.36s^2+7.04s+1} \\ \frac{3}{10.89s^2+4.62s+1} & \frac{10}{36s^2+12s+1} \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{I}, \quad \mathbf{R} = 0.01\mathbf{I}, \quad p = m = 50$$

$$-1 \leq \mathcal{U}_k \leq 1$$





Regular MPC($m \times n_u = 100$)

SVD of H (b=20)

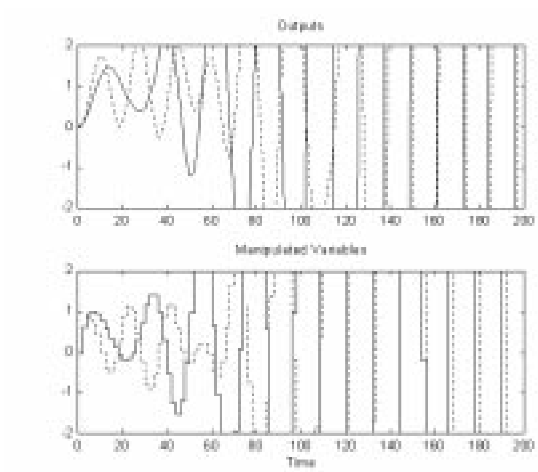
[Ex. 2] Model Error Case

$$G(s) = \begin{bmatrix} \frac{17}{(10s+1)(10s^2+s+1)} & \frac{2.3}{30s+1} \\ \frac{1.3}{20s+1} & \frac{2.8}{(10s+1)(5s^2+s+1)} \end{bmatrix}$$

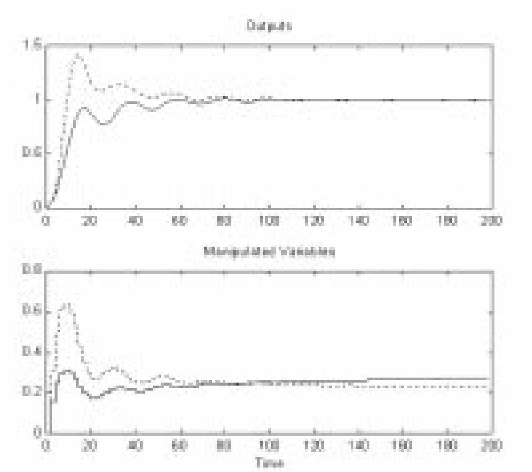
$$G^{model}(s) = \begin{bmatrix} \frac{1.5}{10s+1} & \frac{2.2}{30s+1} \\ \frac{1.2}{20s+1} & \frac{2.6}{10s+1} \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{I}, \quad \mathbf{R} = \mathbf{I}, \quad p = m = 90$$

No constraints



Regular MPC($m \times n_u = 180$)



SVD of H (b=20)