Chapter 2

BASICS OF LINEAR SYSTEMS

2.1 STATE SPACE DESCRIPTION

State Space Model Development

Consider fundamental ODE model:

$$\begin{aligned} \frac{dx_f}{dt} &= f(x_f, u_f) \\ y_f &= g(x_f) \end{aligned}$$

x_f: state vector,

u_f: input vector

y_f: output vector

State Space Model Development (Continued)

$\Downarrow$ linearization w.r.t. an equilibrium $(\bar{x}, \bar{u})$

$$\begin{aligned} \frac{dx}{dt} &= \left(\frac{df}{\partial x_f}\right)_{ss} x + \left(\frac{\partial f}{\partial u_f}\right)_{ss} u \\ y &= \left(\frac{\partial g}{\partial x_f}\right)_{ss} x \end{aligned}$$

where $x = x_f - \bar{x}$, $u = u_f - \bar{u}$.

$\Downarrow$ discretization

$$\begin{aligned} x(k+1) &= Ax(k) + B_u u(k) \\ y(k) &= Cx(k) \end{aligned}$$

State Space Description of Linear Systems

Consider the linear system described by the state equation:

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

Take z-Transformation

$$zX(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z)$$

$$\Downarrow$$

$$Y(z) = C(zI - A)^{-1}BU(z)$$

Solution to Linear System:

$$x(k) = A^n x(0) + \sum_{i=0}^{n-1} A^{n-i-1}Bu(i)$$

Transfer Function

Consider the system described by transfer function:

$$\frac{Y(z)}{U(z)} = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

Then a state space description of the system is

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

where

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$C = [b_1 \; b_2 \; \cdots \; b_{n-1} \; b_n]$$

Transfer Function (Continued)

Example: Consider the transfer function:

$$\frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

Then

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = [0\ 1]$$

Then

$$\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B = [b_1\ b_2] \begin{bmatrix} z + a_1 & +a_2 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [b_1\ b_2] \frac{\begin{bmatrix} z & -a_2 \\ 1 & z + a_1 \end{bmatrix}^{-1}}{z^2 + a_1 z + a_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

Nonuniqueness of State Space Representation

Consider a transfer function $G(z)$. Suppose the state space description of $G(z)$ is

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

Consider a different coordinate system for the state space defined by

$$w(k) = T^{-1}x(k)$$

$$\Downarrow$$

$$w(k+1) = T^{-1}ATw(k) + T^{-1}Bu(k)$$

$$y(k) = CTw(k)$$

Then the transfer function of this system is

$$\frac{Y(z)}{U(z)} = CT(zI - T^{-1}AT)^{-1}T^{-1}B = CT[T^{-1}(zI - A)T]^{-1}T^{-1}B$$

$$= CTT^{-1}(zI - A)^{-1}TT^{-1}B = C(zI - A)^{-1}B = G(z)$$

There exist a multitude of state space representations of a system because there is a multiple infinity coordinate systems of the state space.

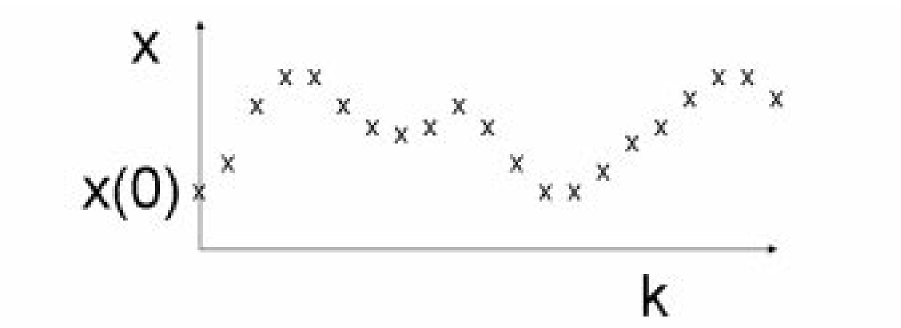
Definition of States

Given a time instant k, the state of the system is the minimal information that are necessary to calculate the future response.

For difference equations, the concept of the state is the same as that of the initial condition.

$$\Downarrow$$

$$\text{State} = x(k)$$



Stability of Linear Systems

A state x is stable if

$$\lim_{n\to\infty} A^n x = 0$$

A linear system

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

is said to be stable if, for all $x \in \mathbf{R}^n$,

$$\lim_{n\to\infty} A^n x = 0$$

$$\Updownarrow$$

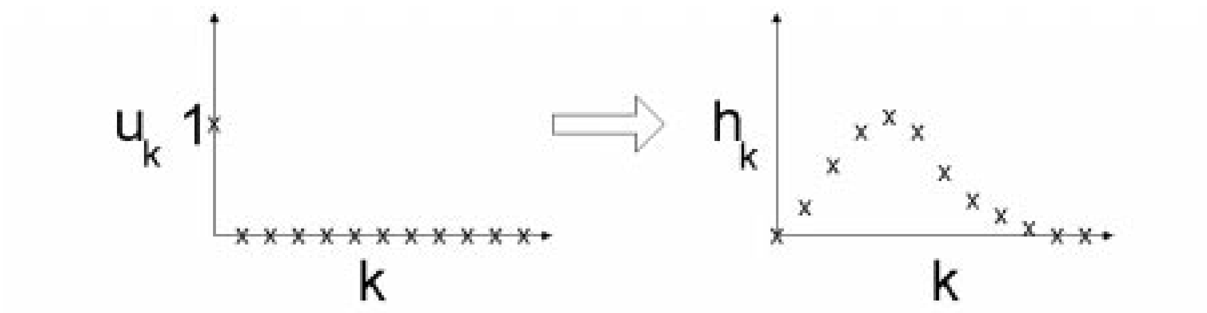
$$\max_i |\lambda_i(A)| < 1$$

2.2 FINITE IMPULSE RESPONSE MODEL

Impulse Responses of Linear Systems

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-i-1} B u(i)$$

Impulse Response Sequence $\{h(k)\}$: $\{y(k)\}$ when $x(0) = 0$ and $u(i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$.



$$\{h(i) = CA^i B\}_{i=0}^{\infty}$$

$$\Downarrow$$

$$y(k) = h(k)x(0) + \sum_{i=0}^{k-1} h(k-i-1)u(i)$$

Impulse Responses of Linear Systems (Continued)

If linear system is stable,

$$\sum_{i=0}^{\infty} \|h(i)\| = \sum_{i=0}^{\infty} \|CA^iB\| < \infty$$

$$\Updownarrow$$

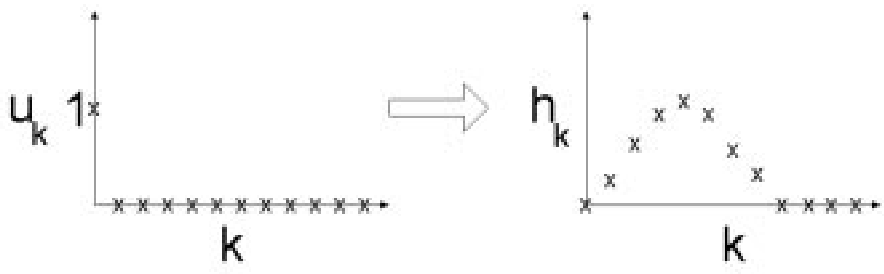
$$\{h(i)\}_{i=0}^{\infty} = \{CA^iB\}_{i=0}^{\infty} \in \ell_1$$

where ℓ_1 is the set of all absolutely summable sequences

$$\Downarrow$$

$$\lim_{i\to\infty} \|h(i)\| = \lim_{i\to\infty} \|CA^iB\| = 0$$

Finite Impulse Response Models



Finite Impulse Response (FIR) Model: Model for which there exists N such that

$$h(i) = 0 \quad \forall i \geq N$$

$$\Downarrow$$

$$y(k) = \sum_{i=1}^{N} h(i)u(k-i)$$

$$\Downarrow$$

FIR model is also called moving average model.

$$\Downarrow$$

Need to store n past inputs: $(u(i-1), \cdots, u(i-N))$

For stable linear systems, $h(i) \to 0$ as $i \to \infty$.

$$\Downarrow$$

FIR model is a good approximation of a stable linear system for large enough N.