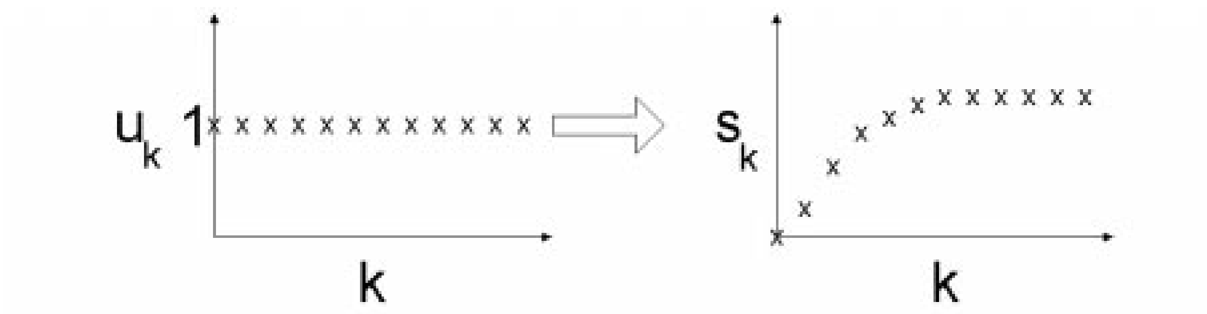
2.3 TRUNCATED STEP RESPONSE MODEL

Step Responses of Linear Systems

$$y(k) = h(k)x(0) + \sum_{i=0}^{k-1} h(k-i-1)u(i)$$

Step Response Sequence $\{s(k)\}$: $\{y(k)\}$ when $x(0) = 0$ and $u(i) = 1$, $i = 0, 1, 2, \cdots$.



Relationship between impulse and step responses:

$$s(k) = \sum_{i=1}^{k} h(i)$$

$$\Updownarrow$$

$$h(k) = s(k) - s(k-1)$$

Truncated Step Response Models

Truncated Step Response (TSR) Model: FIR model represented by its step responses.

$$
\begin{aligned}
y(k) = \sum_{i=1}^{N} h(i)u(k-i) &= \sum_{i=1}^{N} s(i) - s(i-1)u(k-i) \\
&= \sum_{i=1}^{N} s(i)u(k-i) - \sum_{i=1}^{N-1} s(i)u(k-i-1) \\
&= \sum_{i=1}^{N-1} s(i)\Delta u(k-i) + s(N)u(k-N)
\end{aligned}
$$

Truncated Step Response Models (Continued)

Let

$$\tilde{Y}(k) := \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix}$$

when $\Delta u(k) = \Delta u(k+1) = \cdots = 0$. Then

$$\tilde{Y}(k) := \begin{bmatrix} \Sigma_{i=1}^{N-1} s(i)\Delta u(k-i) + s(N)u(k-N) \\ \Sigma_{i=2}^{N-1} s(i)\Delta u(k+1-i) + s(N)u(k-N+1) \\ \Sigma_{i=3}^{N-1} s(i)\Delta u(k+2-i) + s(N)u(k-N+2) \\ \vdots \\ s(N-1)\Delta u(k-1) + s(N)u(k-2) \\ s(N)u(k-1) \end{bmatrix}$$

$$\tilde{Y}(k+1) := \begin{bmatrix} \Sigma_{i=1}^{N-1} s(i)\Delta u(k+1-i) + s(N)u(k-N+1) \\ \Sigma_{i=2}^{N-1} s(i)\Delta u(k+2-i) + s(N)u(k-N+2) \\ \Sigma_{i=3}^{N-1} s(i)\Delta u(k+3-i) + s(N)u(k-N+3) \\ \vdots \\ s(N-1)\Delta u(k) + s(N)u(k-1) \\ s(N)u(k) \end{bmatrix}$$

$$\Downarrow$$

$$\tilde{Y}(k+1) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tilde{Y}(k) + \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(N-1) \\ s(N) \end{bmatrix} \Delta u(k)$$

2.4 REACHABILITY AND OBSERVABILITY

Reachability

A state x is reachable if it can be reached from the zero state in some finite number of times by an appropriate input.

$$\Updownarrow$$

For some n and some $\{u(i)\}$,

$$x(0) = 0$$

$$x(k+1) = Ax(k) + Bu(k), \qquad 0 \leq k \leq n-1$$

$$x(n) = x$$

or

$$x = \sum_{i=0}^{n-1} A^{n-i-1}Bu(i) = Bu(n-1) + \cdots + A^{n-1}Bu(0)$$

A linear system

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

is said to be reachable if any state in the state space is reachable.

$$\Updownarrow$$

$$W_c := [B \ AB \ \cdots \ A^{n-1}B] \text{ has } n \text{ linearly independent columns}$$

Observability

Question: Given A, B, C, D and $\{u(i), y(i)\}_{i=1}^{n}$, can we determine the state $x(1)$ from this data?

$$y(i) = CA^{i-1}x(1) + \sum_{k=1}^{i-1} A^{i-k-1}Bu(k)$$

Define

$$\tilde{y}(i) = y(i) - \sum_{k=1}^{i-1} A^{i-k-1}Bu(k) = CA^{i-1}x(1)$$

$$\Downarrow$$

$$\begin{bmatrix} \tilde{y}(1) \\ \tilde{y}(2) \\ \vdots \\ \tilde{y}(n) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(1)$$

Observability (Continued)

A state x is observable if it is a unique solution of

$$\begin{bmatrix} \tilde{y}(1) \\ \tilde{y}(2) \\ \vdots \\ \tilde{y}(n) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x$$

such that

$$y(i) = CA^{i-1}x + \sum_{k=1}^{i-1} A^{i-k-1}Bu(k)$$

A linear system

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

is said to be observable if any state in the state space is observable.

$$\Updownarrow$$

$$W_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has } n \text{ linearly independent rows}$$

2.5 STATIC STATE FEEDBACK CONTROLLER AND STATE ESTIMATOR

Linear Static State Feedback (Pole Placement)

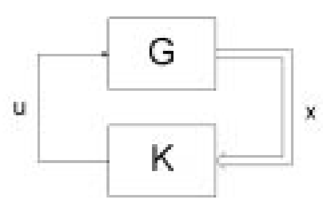
Consider a linear system

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

Let $\{s_i\}_{i=1}^n$ be the set of desired closed loop poles and

$$P(z) = (z - s_1)(z - s_2)\cdots(z - s_n) = z^n + p_1 z^{n-1} + \cdots + p_n$$



Question (Pole Placement Problem): Does there exist linear static state feedback controller $u = Kx$ such that the characteristic polynomial for the closed loop system

$$x(k+1) = (A + BK)x(k)$$

is $P(z)$?

Linear Static State Feedback (Continued)

Suppose there exists T such that $z = Tx$ leads to controllable canonical form:

$$z(k+1) = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} z(k) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$\Downarrow$$

Characteristic polynomial:

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

If

$$u = -\bar{L}z$$

where

$$\bar{L} = [p_1 - a_1 \; p_2 - a_2 \; \cdots \; p_n - a_n]$$

$$\Downarrow$$

$$z(k+1) = \begin{bmatrix} -p_1 & -p_2 & \cdots & -p_{n-1} & -p_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} z(k)$$

$$\Downarrow$$

Closed loop characteristic polynomial:

$$z^n + p_1 z^{n-1} + \cdots + p_n = 0$$

Linear Static State Feedback (Continued)

Question: When does there exist such T?

Let

$$W_c := [B \; AB \; \cdots \; A^{n-1}B]$$

Then

$$\bar{W}_c := [TB \; (TAT^{-1})TB \; \cdots \; (TAT^{-1})^{n-1}TB]$$

$$= [TB \; TAB \; \cdots \; TA^{n-1}B] = TW_c$$

$$\Downarrow$$

If W_c is invertible,

$$T = \bar{W}_c W_c^{-1}$$

Theorem: Pole placement is possible iff the system is reachable.

The pole placing contoller is

$$u = -\bar{L}\bar{W}_c W_c^{-1} x$$

Linear Observer

Consider a linear system

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

Suppose the states are not all measurable and y is only available.

Question: Can we design the state estimator such that the state estimate converges to the actual state?

Given the state estimate of $x(k)$ at $k-1$, $\hat{x}(k|k-1)$,

$$y(k) \neq C\hat{x}(k|k-1)$$

due to the estimation error.

$$\Downarrow$$

$$\hat{x}(k+1|k) = \underbrace{A\hat{x}(k|k-1) + Bu(k)}_{\text{prediction based on the model}} + \underbrace{K[y(k) - C\hat{x}(k|k-1)]}_{\text{correction based on the error}}$$

Define the estimation error as

$$\tilde{x} := x - \hat{x}$$

$$\Downarrow$$

$$\tilde{x}(k+1|k) = A\tilde{x}(k|k-1) - K[y(k) - C\hat{x}(k|k-1)]$$

$$= A\tilde{x}(k|k-1) - K[Cx(k) - C\hat{x}(k|k-1)] = [A - KC]\tilde{x}(k|k-1)$$

Linear Observer (Continued)

Question: Does there exist K such that the characteristic polynomial of $x(k+1) = (A + KC)x(k)$ is the desired polynomial $P(z)$?

$$\Updownarrow$$

Does there exist linear static state feedback controller $v = K^T z$ for the system

$$z(k+1) = A^T z(k) + C^T v(k)$$

such that the characteristic polynomial for the closed loop system

$$z(k+1) = (A^T + C^T K^T) z(k)$$

is the desired polynomial $P(z)$?

From pole placement, we know that this is possible iff

$$[C^T \; A^T C^T \; \cdots \; (A^T)^{n-1} C^T] =: W_o^T$$

is invertible and

$$K = -W_o^{-1} \bar{W}_o \bar{K}$$

where

$$\bar{W}_o = W_o T^T$$

$$\bar{K} = \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \\ \vdots \\ p_n - a_n \end{bmatrix}$$